THE BOCHNER-TYPE FORMULA AND THE FIRST EIGENVALUE OF THE SUB-LAPLACIAN ON A CONTACT RIEMANNIAN MANIFOLD

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ABSTRACT. Contact Riemannian manifolds, with not necessarily integrable complex structures, are the generalization of pseudohermitian manifolds in CR geometry. The Tanaka-Webster-Tanno connection on such a manifold plays the role of Tanaka-Webster connection in the pseudohermitian case. We prove the contact Riemannian version of the pseudohermitian Bochner-type formula, and generalize the CR Lichnerowicz theorem about the sharp lower bound for the first nonzero eigenvalue of the sub-Laplacian to the contact Riemannnian case.

1. INTRODUCTION

Lichnerowicz [20] obtained a sharp lower bound for the first eigenvalue of the Laplacian-Beltrami operator on a compact Riemannian manifold with a lower Ricci bound, and Obata [22] characterized the case of equality. On a pseudohermitian manifold, the sub-Laplacian is the counterpart of the Laplacian-Beltrami operator. The CR analogue of the Lichnerowicz theorem states that for a $(2n + 1)$-dimensional pseudohermitian manifold, $n \geq 3$, satisfying

$$Ric(X, \overline{X}) + \frac{n+1}{2}Tor(X, X) \geq \kappa h(X, \overline{X}), \tag{1.1}$$

the first nonzero eigenvalue of the sub-Laplacian is greater than or equal to $n\kappa/(n + 1)$. This result was first proved by Greenleaf [13]. But due to a mistake in calculation pointed out in [6] and [12], the coefficient $\frac{n+1}{2}$ in (1.1) was mistaken to be $\frac{n}{2}$. The corresponding results for $n = 2$ and $n = 1$ were obtained later in [18] and [8], respectively. The CR Obata-type theorem was conjectured in [6], which states that if $n\kappa/(n + 1)$ is an eigenvalue of the sub-Laplacian on a pseudohermitian manifold, then it is the standard CR structure on the unit sphere in $\mathbb{C}^{n+1}$. This is proved under some additional conditions (cf. [6], [7], [16] and references therein) and without conditions in [19]. There is also a quaternionic contact version of Lichnerowicz theorem [14] (see e.g. [3], [15] and [29] for the quaternionic contact manifolds). In this paper, we generalize the CR Lichnerowiecz theorem to the contact Riemannian case.

A $(2n+1)$-dimensional manifold M is called a *contact manifold* if it has a real 1-form θ, called a *contact form*, such that $\theta \wedge d\theta^n \neq 0$ everywhere on M. There exists a unique vector field T, *the Reeb vector field*, such that $\theta(T) = 1$ and $T \lrcorner d\theta = 0$. It is well known that given a contact manifold (M, θ), there are a Riemannian metric h and a $(1, 1)$-tensor field J on M such that

$$\begin{aligned} h(X, T) &= \theta(X), \\ J^2 &= -Id + \theta \otimes T, \\ d\theta(X, Y) &= h(X, JY), \end{aligned} \tag{1.2}$$

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for any vector fields X and Y (cf. p. 278 in [4]). We call J an *almost complex structure*. Once h is fixed, J is uniquely determined. (M, θ, h, J) is called a *contact Riemannian manifold*.

Let TM be the tangent bundle and $\mathbb{C}TM$ be its complexification. Denote $HM := Ker(\theta)$, the horizontal subbundle. $\mathbb{C}HM$ has a unique subbundle $T^{(1,0)}M$ such that $JX = iX$ for any $X \in \Gamma(T^{(1,0)}M)$. Here and in the following, $\Gamma(S)$ denotes the space of all sections of a vector bundle S. Set $T^{(0,1)}M = \overline{T^{(1,0)}M}$. For any $X \in \Gamma(T^{(0,1)}M)$, we have $JX = -iX$. J is called *integrable* if $\left[\Gamma(T^{(1,0)}M), \Gamma(T^{(1,0)}M)\right] \subset \Gamma(T^{(1,0)}M)$. In particular if J is integrable, J is called a *CR structure* and (M, θ, h, J) is called a *pseudohermitian manifold*. On a contact Riemannian manifold there exists a distinguished connection introduced in [28], called the *Tanaka-Webster-Tanno connection* (or *TWT connection* briefly). In the pseudohermitian case, this connection is exactly the Tanaka-Webster connection. The Tanno tensor is defined as $Q = \nabla J$. J is integrable if and only if the Tanno tensor $Q \equiv 0$ (cf. Proposition 2.1 in [28]). We can also define the sub-Laplacian operator Δ_b. Since there is no obstruction to the existence of the almost complex structure J, contact Riemannian structures exist naturally on any contact manifold and analysis on it has potential applications to the geometry of contact manifolds (cf. [2], [24] and [25] and references therein).

Since the Tanno tensor Q is a (1,2)-tensor, $Q_X := Q(X, \cdot)$ and $\nabla Q(X, X)$ are (1,1)-tensors. Define invariants of contact Riemannian structures:

$$
\begin{aligned}
Q_1(X, X) &= -2Re\big(i \cdot \text{trace}\{Y \longrightarrow \nabla_Y Q(X, X)\}\big), \\
Q_2(X, X) &= \langle Q_X, Q_{\bar{X}}\rangle, \\
Q_3(X, X) &= \text{trace}\{Y \longrightarrow Q_X \circ Q_{\bar{X}}(Y)\},
\end{aligned}
\tag{1.3}
$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $(1,1)$-tensor induced by h, and

$$
Tor(X, X) = 2Re\big(ih(\tau_* X, X)\big),
\tag{1.4}
$$

for any $X \in T^{(1,0)}M$, where τ_* is the *Webster torsion* defined as $\tau_*(X) = \tau(T, X)$, $X \in TM$.

Let ∇u be the gradient of u with respect to the metric h, i.e., $h(\nabla u, X) = Xu$ for any $X \in \Gamma(TM)$. Set $\nabla_H u = \pi_H \nabla u$, where π_H is the orthogonal projection to HM. And $\partial_b u$ is the orthogonal projection of $\nabla_H u$ to $T^{(1,0)}M$. Our main result is as follows.

Theorem 1.1. *On a $(2n+1)$-dimensional contact Riemannian manifold, $n \geq 2$, we have the contact Riemannian Bochner-type formula*

$$
\begin{aligned}
\Delta_b(\|\partial_b u\|^2) = {}& 2\|\nabla^2 u\|^2 - 4\nabla^2 u(T, J\nabla_H u) - 2nTor(\partial_b u, \partial_b u) + 2Ric\left(\partial_b u, \overline{\partial_b u}\right) \\
& + h\left(\nabla_H u, \nabla_H(\Delta_b u)\right) + Q_1(\partial_b u, \partial_b u) - \frac{1}{2}Q_2(\partial_b u, \partial_b u),
\end{aligned}
\tag{1.5}
$$

for any $u \in C_0^\infty(M)$.

Theorem 1.2. *Suppose that on a compact $(2n+1)$-dimensional contact Riemannian manifold, $n \geq 2$, there exists some positive constant κ such that*

$$
Ric(X, \overline{X}) - (n+1)Tor(X, X) + \frac{1}{2}Q_1(X, X) - \frac{2n+7}{8(n-1)}Q_2(X, X) + \frac{3}{2(n-1)}Q_3(X, X) \geq \kappa h(X, \overline{X}),
\tag{1.6}
$$

for any $X \in T^{(1,0)}M$, where Ric is the Ricci tensor. Then the first nonzero eigenvalue λ_1 of Δ_b satisfies

$$\lambda_1 \geq \frac{\kappa n}{n+1}.$$

Note that the coefficients of Tor in (1.5) and (1.6) are different from that (1.1) by a factor -2 (cf. [10] and [19]) . This is because that in our definition (1.2), $d\theta(X,Y) = h(X,JY)$, while in psuedohermitian case, people usually use $d\theta(X,Y) = 2h(JX,Y) = -2h(X,JY)$. When $Q \equiv 0$, Theorem 1.1 and 1.2 coincide with the CR Bochner-type formula and CR Lichnerowiecz theorem, respectively (see e.g. [6], [10], [12], [13], [19]). It is quite interesting to characterize the equality case of (1.6).

In Section 2, we introduce some basic preliminaries, including the TWT connection, the torsion tensor, the curvature tensor and the Tanno tensor. If we choose an orthonormal $T^{(1,0)}M$ frame, there are some simpler relations for the connection coefficients, the Tanno tensor and the structure equations, which will make our calculation easier.

When given an orthonormal $T^{(1,0)}M$ frame, we have $\Gamma^{\bar{\gamma}}_{\alpha\beta} = -\frac{i}{2}Q^{\bar{\gamma}}_{\beta\alpha}$, which vanish in the pseudohermitian case. But in the general case, it may not always vanish. Therefore there exists extra terms involving such connection coefficients in our formulae, e.g. the Bochner-type formula and various integral identities, which will make our calculation more complicated than the pseudohermitian case. The main difficulties of generalizing results to the contact Riemannian case come from handling such extra terms.

In section 3, we introduce the second- and third-order covariant derivatives and their commutation formulae with respect to an orthonormal $T^{(1,0)}M$ frame. In Section 4, we prove the Bochner-type formula on a contact Riemannian manifold. This formula differs from the pseudohermitian case by terms involving the Tanno tensor. And it coincides with the CR Bochner-type formula (cf. e.g. Proposition 9.5 in [10] or Theorem 6 in [19]) when the almost complex structure J is integrable. Similarly to pseudohermitian case, the term $\nabla^2 u(T, J\nabla_H u)$ in the Bochner-type formula can be controlled by using two integral identities. But here, in one identity, we have to use another identity to handle extra terms depending on the Tanno tensor Q. It is done in Section 5. In Section 6, with the preparation above, we prove the main Theorem 1.2.

2. CONNECTION COEFFICIENTS, TORSIONS AND CURVATURES ON CONTACT RIEMANNIAN MANIFOLDS

2.1. TWT connection, the Tanno tensor and the orthonormal $T^{(1,0)}M$ frame.

Proposition 2.1. *(cf. (7)-(10) in [4]) On a contact Riemannian manifold (M, θ, h, J), there exists a unique linear connection such that*

$$\begin{aligned}
&\nabla\theta = 0, \quad \nabla T = 0, \\
&\nabla h = 0, \\
&\tau(X,Y) = 2d\theta(X,Y)T, \quad X, Y \in \Gamma(HM), \\
&\tau(T, JZ) = -J\tau(T, Z), \quad Z \in \Gamma(TM),
\end{aligned} \tag{2.1}$$

where τ is the torsion of ∇.

∇ is called the *TWT connection*. The *Tanno tensor* Q (cf. (10) in [10]) is defined as

$$Q(X,Y) := (\nabla_Y J)X, \quad \text{for } X, Y \in \Gamma(TM). \tag{2.2}$$

We extend h, J and ∇ to the complexified tangent bundle by $\mathbb{C}$-linear extension:

$$h(X_1 + iY_1, X_2 + iY_2) := h(X_1, X_2) - h(Y_1, Y_2) + i\big(h(X_1, Y_2) + h(X_2, Y_1)\big),$$

$$J(X_1 + iY_1) := JX_1 + iJY_1,$$

$$\nabla_{(X_1+iY_1)}(X_2 + iY_2) := \nabla_{X_1} X_2 - \nabla_{Y_1} Y_2 + i\big(\nabla_{X_1} Y_2 + \nabla_{Y_1} X_2\big).$$

for any $Z_j = X_j + iY_j \in \mathbb{C}TM$, $j = 1, 2$.

Proposition 2.2. *Let $W_0 := T$, the Reeb vector. We can choose a local $T^{(1,0)}M$-frame $\{W_j\} = \{W_a, W_0\} = \{W_\alpha, W_{\bar\alpha}, T\}$ with $W_\alpha \in T^{(1,0)}M$, $W_{\bar\alpha} = \overline{W_\alpha} \in T^{(0,1)}M$ on a neighborhood U such that*

$$h_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}; \qquad h_{\bar\alpha\beta} = \delta_{\bar\alpha\beta} = \delta_{\alpha\bar\beta}; \qquad h_{\alpha\beta} = 0.$$

We call this frame an orthonormal $T^{(1,0)}M$-frame.

Proof. Note that (1.2) leads to

$$\begin{aligned} & JT = 0, \quad \theta(JX) = 0, \\ & h(X, Y) = h(JX, JY) + \theta(X)\theta(Y), \quad d\theta(X, JY) = -d\theta(JX, Y), \end{aligned} \tag{2.3}$$

for any $X, Y \in TM$ (cf. p. 351 in [28]). Choose a vector field X_1 in $\Gamma(HM)$ such that $h(X_1, X_1) = \frac{1}{2}$ and let $X_{n+1} := JX_1$. Then $h(X_1, X_{n+1}) = h(X_1, JX_1) = d\theta(X_1, X_1) = 0$, i.e. X_{n+1} is automatically orthogonal to X_1, and by third identity in (2.3), we get $h(X_{n+1}, X_{n+1}) = h(JX_1, JX_1) = h(X_1, X_1) = \frac{1}{2}$. We choose X_2 orthogonal to $span\{X_1, JX_1\}$, and define $X_{n+2} := JX_2$. Repeating the procedure, we find a local orthogonal basis $X_1, \cdots, X_{2n}$ with $h(X_a, X_b) = \frac{1}{2}\delta_{ab}$ and $JX_\alpha = X_{\alpha+n}$. Now define

$$W_\alpha := X_\alpha - iX_{\alpha+n}, \qquad W_{\bar\alpha} := \overline{W_\alpha}. \tag{2.4}$$

It is direct to see that $JW_\alpha = iW_\alpha$ and $JW_{\bar\alpha} = -iW_{\bar\alpha}$. Namely, $W_\alpha \in T^{(1,0)}M$ and $W_{\bar\alpha} \in T^{(0,1)}M$. Then by Remark 2.1 for the complex extension we get $h(W_\alpha, W_\beta) = h(X_\alpha - iX_{\alpha+n}, X_\beta - iX_{\beta+n}) = 0$, $h_{\bar\alpha\bar\beta} = 0$ and $h(W_\alpha, W_{\bar\beta}) = \delta_{\alpha\bar\beta}$, $h_{\bar\alpha\beta} = \delta_{\bar\alpha\beta} = \delta_{\alpha\bar\beta}$. $\qquad\square$

Remark 2.1. (1) For the multi-index, we adopt the following index conventions in this paper.

$$\alpha, \beta, \gamma, \rho, \lambda, \mu, \cdots \in \{1, \cdots, n\}, \qquad a, b, c, d, e, \cdots \in \{1, 2, \cdots, 2n\},$$

$$j, k, l, r, s, \cdots \in \{0, 1, \cdots, 2n\}, \qquad \bar\alpha = \alpha + n.$$

(2) In this paper, the Einstein summation convention will be used. Moreover, if indices α and $\bar\alpha$ both appear in low (or upper) indices, then the index α will be taken summation, e.g. $h_{\alpha\bar\alpha} = \sum_\alpha h_{\alpha\bar\alpha}$.

From now on, we choose a local orthonormal $T^{(1,0)}M$-frame $\{W_j\}$. In particular, by (1.2), $h(T, W_a) = h(W_a, T) = \theta(W_a) = 0$ and $h(T, T) = \theta(T) = 1$. We denote $h_{ab} = h(W_a, W_b)$ and use h_{ab} and its inverse matrix to lower and raise indices. Let $\{\theta^\beta, \theta^{\bar\beta}, \theta\}$ denote the dual coframe to $\{W_\alpha, W_{\bar\alpha}, T\}$, i.e., $\theta^\beta(W_\alpha) = \delta_\alpha^\beta$, $\theta^\beta(W_{\bar\alpha}) = \theta^\beta(T) = 0$, $\theta^{\bar\beta} := \overline{\theta^\beta}$, and $\theta(W_\alpha) = \theta(W_{\bar\alpha}) = 0$, $\theta(T) = 1$. Set $\theta^0 := \theta$. The connection 1-form with respect to $\{W_j\}$ is given by $\nabla W_j = \omega_j^k \otimes W_k$, and set $\omega_j^k := \Gamma_{ij}^k \theta^i$, i.e. $\nabla_{W_i} W_j = \Gamma_{ij}^k W_k$. By (2.1), we get $\theta(\nabla X) = h(\nabla X, T) = d(h(X, T)) - h(X, \nabla T) = 0$ for any $X \in TM$, namely $\Gamma_{ij}^0 = 0$. And $\nabla T = 0$ implies $\Gamma_{i0}^k = 0$.

By the dual argument, we have

$$\nabla_{W_i} \theta^k = -\Gamma_{ij}^k \theta^j.$$

And for any (r,s)-tensor φ with components $\varphi^{k_1\cdots k_r}{}_{j_1\cdots j_s} = \varphi(\theta^{k_1}, \cdots, \theta^{k_r}, W_{j_1}, \cdots, W_{j_s})$, covariant derivatives of φ are given by

$$\varphi^{k_1\cdots k_r}{}_{j_1\cdots j_s,i} = W_i(\varphi^{k_1\cdots k_r}{}_{j_1\cdots j_s}) + \sum_{t=1}^{r}\Gamma^{k_t}_{il}\varphi^{k_1\cdots l\cdots k_r}{}_{j_1\cdots j_s} - \sum_{t=1}^{s}\Gamma^l_{ij_t}\varphi^{k_1\cdots k_r}{}_{j_1\cdots l\cdots j_s}. \tag{2.5}$$

Denote the components of the almost complex structure J by $J^l{}_j$. We write $Q(W_j, W_k) = (\nabla_{W_k}J)W_j = Q^l_{jk}W_l$. Equivalently, $Q^l_{jk} := J^l{}_{j,k}$. Applying (2.5), we have

$$\begin{aligned}
Q^l_{jk} &= W_k J^l{}_j + \Gamma^l_{ks}J^s{}_j - \Gamma^s_{kj}J^l{}_s, \\
Q^l_{jk,s} &= W_s Q^l_{jk} - \Gamma^r_{sj}Q^l_{rk} - \Gamma^r_{sk}Q^l_{jr} + \Gamma^l_{sr}Q^r_{jk}.
\end{aligned} \tag{2.6}$$

Proposition 2.3. *(cf. (16)-(18) in [4]) With respect to a local $T^{(1,0)}$-frame $\{W_j\}$, the components of tensor Q has the following property:*

$$\begin{aligned}
&Q^\gamma_{\beta\alpha} = 0, \quad \Gamma^\gamma_{\alpha\bar\beta} = 0, \quad Q^\gamma_{\bar\beta\alpha} = 0, \quad Q^{\bar\gamma}_{\bar\beta\alpha} = 0, \quad \Gamma^{\bar\gamma}_{\alpha\beta} = -\frac{i}{2}Q^{\bar\gamma}_{\beta\alpha}, \\
&\Gamma^0_{\alpha\bar\beta} = 0, \quad \Gamma^0_{\alpha\beta} = 0, \quad \Gamma^\gamma_{\alpha 0} = 0, \quad \Gamma^{\bar\gamma}_{\alpha 0} = 0 \quad \Gamma^{\bar\gamma}_{0\beta} = 0, \quad \Gamma^0_{0\beta} = 0, \\
&Q^k_{0j} = 0, \quad Q^k_{i0} = 0, \quad Q^0_{ij} = 0.
\end{aligned} \tag{2.7}$$

In particular, only the components $Q^{\bar\gamma}_{\beta\alpha}$ of tensor Q are non-vanishing. In (2.7), $Q^k_{0j} = 0$ follows from $Q^k_{0j}W_k = Q(T, W_j) = (\nabla_{W_j}J)T = \nabla_{W_j}(JT) - J\nabla_{W_j}T \equiv 0$ by (2.1) and (2.3) for any W_j. And $Q^k_{i0} = 0$ follows from setting $X = W_i$, $Y = T$, $Z = W_l$ in identity (cf. (15) in [4]):

$$2h(Q(X,Y),Z) = h(N^{(1)}(X,Z) - \theta(X)N^{(1)}(T,Z) + \theta(Z)N^{(1)}(X,T), JY), \tag{2.8}$$

for any $X, Y, Z \in TM$, where

$$N^{(1)} = [J,J] + 2(d\theta)\otimes T, \quad [J,J](X,Y) = J^2[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY],$$

to get $Q^k_{i0}h_{kl} \equiv 0$ by $JT \equiv 0$ in (2.3). For $Q^0_{ij} = 0$, since we already have $Q^0_{i0} = Q^0_{0j} = 0$, it remains to prove $Q^0_{ab} = 0$. This follows from $Q^0_{ab} = \theta\left(Q(W_a, W_b)\right) = \theta\left((\nabla_{W_b}J)W_a\right) = h(T, (\nabla_{W_b}J)W_a) = h(T, \nabla_{W_b}(JW_a)) - h(T, J\nabla_{W_b}W_a) = \theta\left(\nabla_{W_b}(JW_a)\right) - \theta(J\nabla_{W_b}W_a) = 0$ by W_a, W_b being horizontal.

Remark 2.2. In pseudohermitian case, $\Gamma^{\bar\gamma}_{\alpha\beta} = 0$ by the Tanaka-Webster connection preserving $T^{(1,0)}M$. But in general case, $\Gamma^{\bar\gamma}_{\alpha\beta} = -\frac{i}{2}Q^{\bar\gamma}_{\beta\alpha}$ may not vanish.

Recall that only the components $Q^{\bar\gamma}_{\beta\alpha}$ of tensor Q are non-vanishing. So by definition (1.3), with respect to a local orthonormal $T^{(1,0)}$-frame $\{W_j\}$, we have

$$\begin{aligned}
Q_1(X,X) &= -2Re\left(i\cdot\text{trace}\{W_j\longrightarrow\nabla_{W_j}Q(X,X)\}\right) = -2Re\left(iQ^j_{\alpha\beta,j}X^\alpha X^\beta\right) \\
&= -2Re\left(iQ^{\bar\gamma}_{\alpha\beta,\bar\gamma}X^\alpha X^\beta\right) = i\left(Q^\gamma_{\bar\alpha\bar\beta,\gamma}X^{\bar\alpha}X^{\bar\beta} - Q^{\bar\gamma}_{\alpha\beta,\bar\gamma}X^\alpha X^\beta\right), \\
Q_2(X,X) &= h_{ij}h^{kl}Q^i_{\alpha k}Q^j_{\bar\beta l}X^\alpha X^{\bar\beta} = h_{\lambda\bar\rho}h^{\gamma\bar\mu}Q^{\bar\rho}_{\alpha\gamma}Q^\lambda_{\bar\beta\bar\mu}X^\alpha X^{\bar\beta} = Q^{\bar\rho}_{\alpha\gamma}Q^\rho_{\bar\beta\bar\gamma}X^\alpha X^{\bar\beta}, \\
Q_3(X,X) &= \text{trace}\{W_j\longrightarrow Q_{W_\alpha}\circ Q_{W_{\bar\beta}}(W_j)X^\alpha X^{\bar\beta}\} \\
&= \text{trace}\{W_j\longrightarrow Q^i_{\bar\beta j}Q_{W_\alpha}(W_i)X^\alpha X^{\bar\beta}\} = Q^j_{\alpha i}Q^i_{\bar\beta j}X^\alpha X^{\bar\beta} = Q^{\bar\gamma}_{\alpha\rho}Q^\rho_{\bar\beta\bar\gamma}X^\alpha X^{\bar\beta},
\end{aligned} \tag{2.9}$$

for any $X = X^\alpha W_\alpha \in T^{(1,0)}M$. According to (2.6) and $\Gamma^\gamma_{\alpha\bar\beta} = 0$ in (2.7), we get the components

$$Q^{\bar\gamma}_{\alpha\beta,\bar\rho} = W_{\bar\rho} Q^{\bar\gamma}_{\alpha\beta} - \Gamma^e_{\bar\rho\alpha} Q^{\bar\gamma}_{e\beta} - \Gamma^e_{\bar\rho\beta} Q^{\bar\gamma}_{\alpha e} + \Gamma^{\bar\gamma}_{\bar\rho e} Q^e_{\alpha\gamma} = W_{\bar\rho} Q^{\bar\gamma}_{\alpha\beta} - \Gamma^\mu_{\bar\rho\alpha} Q^{\bar\gamma}_{\mu\beta} - \Gamma^\mu_{\bar\rho\beta} Q^{\bar\gamma}_{\alpha\mu} + \Gamma^{\bar\gamma}_{\bar\rho\bar\mu} Q^{\bar\mu}_{\alpha\beta}, \quad (2.10)$$

of tensor ∇Q.

Proposition 2.4. *With respect to a local orthonormal $T^{(1,0)}M$-frame $\{W_j\}$, we have*

$$\begin{aligned}
\Gamma^\gamma_{\alpha\beta} &= -\Gamma^{\bar\beta}_{\alpha\bar\gamma}, \qquad \Gamma^{\bar\gamma}_{\alpha\beta} = -\Gamma^{\bar\beta}_{\alpha\gamma}, \\
Q^{\bar\gamma}_{\beta\alpha} &= -Q^{\bar\beta}_{\gamma\alpha}, \qquad Q^{\bar\gamma}_{\alpha\beta,\bar\rho} = -Q^{\bar\alpha}_{\gamma\beta,\bar\rho}, \\
Q^{\bar\gamma}_{\alpha\beta} &= Q^{\bar\beta}_{\alpha\gamma} - Q^{\bar\beta}_{\gamma\alpha},
\end{aligned} \qquad (2.11)$$

and their conjugation.

Proof. By (2.7) and Proposition 2.2, we have

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\rho_{\alpha\beta}\delta_{\rho\bar\gamma} = h(\nabla_{W_\alpha} W_\beta, W_{\bar\gamma}) = W_\alpha(h_{\beta\bar\gamma}) - h(W_\beta, \nabla_{W_\alpha} W_{\bar\gamma}) = -h_{\beta\bar\mu}\Gamma^{\bar\mu}_{\alpha\bar\gamma} = -\Gamma^{\bar\beta}_{\alpha\bar\gamma}.$$

$\Gamma^{\bar\gamma}_{\alpha\beta} = -\Gamma^{\bar\beta}_{\alpha\gamma}$ follows similarly. Then we get $Q^{\bar\gamma}_{\beta\alpha} = 2i\Gamma^{\bar\gamma}_{\alpha\beta} = -2i\Gamma^{\bar\beta}_{\alpha\gamma} = -Q^{\bar\beta}_{\gamma\alpha}$ by (2.7). The fourth identity in (2.11) follows from this identity.

For the last identity in (2.11), setting $X = W_\alpha$, $Y = W_\beta$, $Z = W_\gamma$ in (2.8), we get

$$\begin{aligned}
2h(Q(W_\alpha, W_\beta), W_\gamma) &= h([J,J](W_\alpha, W_\gamma), JW_\beta) \\
&= ih(J^2[W_\alpha, W_\gamma] + [JW_\alpha, JW_\gamma] - J[JW_\alpha, W_\gamma] - J[W_\alpha, JW_\gamma], W_\beta) \\
&= -2ih([W_\alpha, W_\gamma], W_\beta) + 2h(J[W_\alpha, W_\gamma], W_\beta) = -4ih([W_\alpha, W_\gamma], W_\beta).
\end{aligned}$$

By the definition of the torsion tensor and (2.7), the $T^{(0,1)}M$-components of $[W_\alpha, W_\gamma]$ is given by

$$\begin{aligned}
[W_\alpha, W_\gamma] &= \nabla_{W_\alpha} W_\gamma - \nabla_{W_\gamma} W_\alpha - \tau(W_\alpha, W_\gamma) = \Gamma^{\bar\rho}_{\alpha\gamma} W_{\bar\rho} - \Gamma^{\bar\rho}_{\gamma\alpha} W_{\bar\rho} \\
&= -\frac{i}{2} Q^{\bar\rho}_{\gamma\alpha} W_{\bar\rho} + \frac{i}{2} Q^{\bar\rho}_{\alpha\gamma} W_{\bar\rho} \quad mod \quad W_\rho, \quad T.
\end{aligned} \qquad (2.12)$$

Therefore, we get

$$2Q^{\bar\mu}_{\alpha\beta} h_{\gamma\bar\mu} = 2h(Q(W_\alpha, W_\beta), W_\gamma) = -4ih([W_\alpha, W_\gamma], W_\beta) = -2Q^{\bar\rho}_{\gamma\alpha} h_{\beta\bar\rho} + 2Q^{\bar\rho}_{\alpha\gamma} h_{\beta\bar\rho}.$$

The last identity of (2.11) holds. $\qquad\qquad\square$

2.2. The Webster torsion, the curvature tensor and the structure equations.

Lemma 2.1. *(cf. Lemma 1 in [4]) The Webster torsion has following properties:*

$$\tau_*(T) = 0; \quad \tau_* T_{(1,0)} M \subseteq T_{(0,1)} M; \quad \tau_* T_{(1,0)} M \subseteq T_{(1,0)} M.$$

By Lemma 2.1, we can write $\tau_*(W_\alpha) = A^{\bar\beta}_\alpha W_{\bar\beta}$. Set $\tau^\alpha := A^\alpha_{\bar\beta} \theta^{\bar\beta}$. So by (1.4), with respect to $\{W_j\}$, we have

$$Tor(X, X) = 2Re\left(iA_{\alpha\beta} X^\alpha X^\beta\right) = iA_{\alpha\beta} X^\alpha X^\beta - iA_{\bar\alpha\bar\beta} X^{\bar\alpha} X^{\bar\beta}, \qquad (2.13)$$

for any $X = X^\alpha W_\alpha \in T^{(1,0)}M$.

The components $R_j{}^s{}_{kl}$ of the curvature tensor $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is given by $R(W_k, W_l)W_j = R_j{}^s{}_{kl}W_s$. The Ricci tensor is given by

$$Ric(Y, Z) = \text{trace}\{X \longrightarrow R(X, Z)Y\},$$

for any $X, Y, Z \in TM$ (cf. p. 299 in [4]). And the scalar curvature is $R = trace(Ric)$. With respect to a $T^{(1,0)}M$-frame, $R_{\alpha\bar{\beta}} = R_\alpha{}^\gamma{}_{\gamma\bar{\beta}}$ (cf. (53) in [4]). The scalar curvature is $R = h^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}$.

Proposition 2.5. *(cf. (13), (14) and (39) in [4]) With respect to a local orthonormal $T^{(1,0)}M$-frame $\{W_j\}$, we have the following structure equations:*

$$d\theta = -2ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} = -2i\theta^\alpha \wedge \theta^{\bar{\alpha}},$$

$$d\theta^\alpha = \theta^b \wedge \omega_b^\alpha + \theta \wedge \tau^\alpha = \theta^b \wedge \omega_b^\alpha + A_{\bar{\beta}}^\alpha \theta \wedge \theta^{\bar{\beta}},$$

$$d\omega_a^b - \omega_a^c \wedge \omega_c^b = R_a{}^b{}_{\lambda\bar{\mu}}\theta^\lambda \wedge \theta^{\bar{\mu}} + \frac{1}{2}R_a{}^b{}_{\lambda\mu}\theta^\lambda \wedge \theta^\mu + \frac{1}{2}R_a{}^b{}_{\bar{\lambda}\bar{\mu}}\theta^{\bar{\lambda}} \wedge \theta^{\bar{\mu}} + R_a{}^b{}_{0\bar{\mu}}\theta \wedge \theta^{\bar{\mu}} - R_a{}^b{}_{\lambda 0}\theta \wedge \theta^\lambda$$

$$R(X, Y)W_a = 2\left(d\omega_a^b - \omega_a^c \wedge \omega_c^b\right)(X, Y)W_b.$$

$$(2.14)$$

Here following [4] we use the following definition for exterior product and exterior derivatives

$$\phi \wedge \psi(X, Y) = \frac{1}{2}\left(\phi(X)\psi(Y) - \psi(X)\phi(Y)\right),$$

$$2(d\phi)(X, Y) = X(\phi(Y)) - Y(\phi(X)) - \phi([X, Y]) = (\nabla_X \phi)Y - (\nabla_Y \phi)X + \phi(\tau(X, Y)),$$

$$(2.15)$$

for any 1-form ϕ and ψ. The second identity in (2.14) follows from the orthonormality of $\{W_a\}$.

Corollary 2.1. *With respect to a local orthonormal $T^{(1,0)}M$-frame $\{W_j\}$, set $J_{ab} = h_{ac}J^c{}_b$. We have*

$$R_a{}^b{}_{cd} = W_c\Gamma_{da}^b - W_d\Gamma_{ca}^b - \Gamma_{cd}^e\Gamma_{ea}^b + \Gamma_{dc}^e\Gamma_{ea}^b - \Gamma_{ca}^e\Gamma_{de}^b + \Gamma_{da}^e\Gamma_{ce}^b + 2\Gamma_{0a}^b J_{cd}. \qquad (2.16)$$

Proof. Note that $h(W_a, JW_b) = h(W_a, J^c{}_b W_c) = h_{ac}J^c{}_b = J_{ab}$. By (2.1) and the last identity in (2.14), we have

$$R_a{}^b{}_{cd} = 2(d\omega_a^b)(W_c, W_d) - 2\omega_a^e \wedge \omega_e^b(W_c, W_d)$$

$$= (\nabla_{W_c}\omega_a^b)(W_d) - (\nabla_{W_d}\omega_a^b)(W_c) + \omega_a^b(\tau(W_c, W_d)) - \Gamma_{ca}^e\Gamma_{de}^b + \Gamma_{da}^e\Gamma_{ce}^b$$

$$= W_c\Gamma_{da}^b - W_d\Gamma_{ca}^b - \omega_a^b(\nabla_{W_c}W_d) + \omega_a^b(\nabla_{W_d}W_c) + \omega_a^b(2h(W_c, JW_d)T) - \Gamma_{ca}^e\Gamma_{de}^b + \Gamma_{da}^e\Gamma_{ce}^b$$

$$= W_c\Gamma_{da}^b - W_d\Gamma_{ca}^b - \Gamma_{cd}^e\Gamma_{ea}^b + \Gamma_{dc}^e\Gamma_{ea}^b - \Gamma_{ca}^e\Gamma_{de}^b + \Gamma_{da}^e\Gamma_{ce}^b + 2\Gamma_{0a}^b J_{cd}.$$

$$\square$$

Proposition 2.6. *For the components of the curvature tensor, we have the following commutation relations:*

$$R_{\alpha\bar{\beta}\gamma\bar{\mu}} = -R_{\alpha\bar{\beta}\bar{\mu}\gamma}, \quad R_{\alpha\bar{\beta}\gamma\bar{\mu}} = -R_{\bar{\beta}\alpha\gamma\bar{\mu}}, \quad R_{\alpha\bar{\beta}\gamma\bar{\mu}} = R_{\gamma\bar{\beta}\alpha\bar{\mu}}, \qquad (2.17)$$

and their conjugation with respect to a local orthonormal $T^{(1,0)}M$-frame $\{W_j\}$.

Proof. The first identity in (2.17) follows directly by the definition of the curvature tensor. For the last one in (2.17), we refer to Corollary 1 in [4]. For the second identity in (2.17), note that $\nabla h = 0$ implies $Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$ and h_{ab} are constants. Then

$$
\begin{aligned}
R_{\alpha\bar\beta\gamma\bar\mu} &= h\big(\nabla_{W_\gamma}\nabla_{W_{\bar\mu}}W_\alpha - \nabla_{W_{\bar\mu}}\nabla_{W_\gamma}W_\alpha - \nabla_{[W_\gamma, W_{\bar\mu}]}W_\alpha, W_{\bar\beta}\big) \\
&= W_\gamma\Big(h\big(\nabla_{W_{\bar\mu}}W_\alpha, W_{\bar\beta}\big)\Big) - h\big(\nabla_{W_{\bar\mu}}W_\alpha, \nabla_{W_\gamma}W_{\bar\beta}\big) - W_{\bar\mu}\Big(h\big(\nabla_{W_\gamma}W_\alpha, W_{\bar\beta}\big)\Big) \\
&\quad + h\big(\nabla_{W_\gamma}W_\alpha, \nabla_{W_{\bar\mu}}W_{\bar\beta}\big) - [W_\gamma, W_{\bar\mu}]\big(h(W_\alpha, W_{\bar\beta})\big) + h(W_\alpha, \nabla_{[W_\gamma, W_{\bar\mu}]}W_{\bar\beta}) \\
&= W_\gamma W_{\bar\mu}\big(h_{\alpha\bar\beta}\big) - W_\gamma\big(h(W_\alpha, \nabla_{W_{\bar\mu}}W_{\bar\beta})\big) - W_{\bar\mu}\big(h(W_\alpha, \nabla_{W_\gamma}W_{\bar\beta})\big) \\
&\quad + h(W_\alpha, \nabla_{W_{\bar\mu}}\nabla_{W_\gamma}W_{\bar\beta}) - W_{\bar\mu}W_\gamma\big(h_{\alpha\bar\beta}\big) + W_{\bar\mu}\big(h(W_\alpha, \nabla_{W_\gamma}W_{\bar\beta})\big) \\
&\quad + W_\gamma\big(h(W_\alpha, \nabla_{W_{\bar\mu}}W_{\bar\beta})\big) - h(W_\alpha, \nabla_{W_\gamma}\nabla_{W_{\bar\mu}}W_{\bar\beta}) - [W_\gamma, W_{\bar\mu}]\big(h_{\alpha\bar\beta}\big) + h(W_\alpha, \nabla_{[W_\gamma, W_{\bar\mu}]}W_{\bar\beta}) \\
&= h(W_\alpha, \nabla_{W_{\bar\mu}}\nabla_{W_\gamma}W_{\bar\beta}) - h(W_\alpha, \nabla_{W_\gamma}\nabla_{W_{\bar\mu}}W_{\bar\beta}) + h(W_\alpha, \nabla_{[W_\gamma, W_{\bar\mu}]}W_{\bar\beta}) = -R_{\bar\beta\alpha\gamma\bar\mu}.
\end{aligned}
$$

$\square$

Remark 2.3. By Remark 2.1 for the complex extension, it's easy to see that under the complex conjugation, the Riemannian metric h, the almost complex structure J, the TWT connection ∇, the torsion tensor A, the curvature tensor R and the Tanno tensor Q are preserved, i.e.,

$$
\overline{h(Z_1, Z_2)} = h(\overline{Z_1}, \overline{Z_2}), \quad \overline{JZ_1} = \overline{J}\,\overline{Z_1}, \quad \overline{\nabla_{Z_1}Z_2} = \nabla_{\overline{Z_1}}\overline{Z_2},
$$

$$
\overline{\tau(Z_1, Z_2)} = \tau(\overline{Z_1}, \overline{Z_2}), \quad \overline{R(Z_1, Z_2)Z_3} = R(\overline{Z_1}, \overline{Z_2})\overline{Z_3}, \quad \overline{Q(Z_1, Z_2)} = Q(\overline{Z_1}, \overline{Z_2}),
$$

for any $Z_1, Z_2, Z_3 \in \mathbb{C}HM$. The complex conjugation can be reflected in the indices of the components of ω_a^b, h_{ab}, J_a^b, A_{ab}, R_{abcd} and their covariant derivatives, e.g.,

$$
\overline{\omega_\alpha^\beta} = \omega_{\bar\alpha}^{\bar\beta}, \quad \overline{J_\alpha^\beta} = J_{\bar\alpha}^{\bar\beta}, \quad \overline{h_{\alpha\bar\beta}} = h_{\bar\alpha\beta}.
$$

3. THE SECOND- AND THIRD-ORDER COVARIANT DERIVATIVES AND THEIR COMMUTATION FORMULAE

3.1. **The second- and third-order covariant derivatives.** The second-order covariant derivative of u is defined as

$$
\nabla^2 u(X, Y) := X(Yu) - (\nabla_X Y)u, \quad u_{jk} := \nabla^2 u(W_j, W_k), \tag{3.1}
$$

for any vector fields X, Y, and the third-order covariant derivative of u is defined as

$$
\nabla^3 u(X, Y, Z) = \big(\nabla_X \nabla^2 u\big)(Y, Z) = X(\nabla^2 u(Y, Z)) - \nabla^2 u(\nabla_X Y, Z) - \nabla^2 u(Y, \nabla_X Z),
$$
$$
u_{jkl} := \nabla^3 u(W_j, W_k, W_l). \tag{3.2}
$$

for any vector fields X, Y, Z. By (3.1), for the second-order covariant derivative, we have

$$
u_{jk} = \nabla^2 u(W_j, W_k) = W_j W_k u - (\nabla_{W_j} W_k)u = W_j(u_k) - \Gamma_{jk}^l u_l.
$$

In particular, by the vanishing of connection coefficients in (2.7) we get

$$
\begin{aligned}
u_{\alpha\lambda} &= \nabla^2 u(W_\alpha, W_\lambda) = W_\alpha(u_\lambda) - \Gamma_{\alpha\lambda}^\beta u_\beta - \Gamma_{\alpha\lambda}^{\bar\beta} u_{\bar\beta} = W_\alpha(u_\lambda) - \Gamma_{\alpha\lambda}^\beta u_\beta + \frac{i}{2}Q_{\lambda\alpha}^{\bar\beta} u_{\bar\beta}, \\
u_{\alpha\bar\lambda} &= W_\alpha(u_{\bar\lambda}) - \Gamma_{\alpha\bar\lambda}^{\bar\beta} u_{\bar\beta}, \\
u_{\alpha 0} &= W_\alpha(u_0), \quad u_{0\alpha} = T(u_\alpha) - \Gamma_{0\alpha}^\beta u_\beta.
\end{aligned} \tag{3.3}
$$

In the following, the vanishing of connection coefficients in (2.7), especially,

$$\Gamma^{\gamma}_{\alpha\bar{\beta}} = \Gamma^{\bar{\gamma}}_{\bar{\alpha}\beta} = 0,$$

will be used frequently. By (3.2), for the third-order covariant derivative, we have

$$u_{abc} = W_a(u_{bc}) - \Gamma^d_{ab}u_{dc} - \Gamma^d_{ac}u_{bd}.$$

In particular, by (2.7), we have

$$
\begin{aligned}
u_{\alpha\beta\gamma} &= W_\alpha(u_{\beta\gamma}) - \Gamma^\mu_{\alpha\beta}u_{\mu\gamma} - \Gamma^{\bar{\mu}}_{\alpha\beta}u_{\bar{\mu}\gamma} - \Gamma^\mu_{\alpha\gamma}u_{\beta\mu} - \Gamma^{\bar{\mu}}_{\alpha\gamma}u_{\beta\bar{\mu}}, \\
u_{\bar{\alpha}\beta\gamma} &= W_{\bar{\alpha}}(u_{\beta\gamma}) - \Gamma^\mu_{\bar{\alpha}\beta}u_{\mu\gamma} - \Gamma^\mu_{\bar{\alpha}\gamma}u_{\beta\mu}, \\
u_{\alpha\bar{\beta}\gamma} &= W_\alpha(u_{\bar{\beta}\gamma}) - \Gamma^{\bar{\mu}}_{\alpha\bar{\beta}}u_{\bar{\mu}\gamma} - \Gamma^\mu_{\alpha\gamma}u_{\bar{\beta}\mu} - \Gamma^{\bar{\mu}}_{\alpha\gamma}u_{\bar{\beta}\bar{\mu}}, \\
u_{\bar{\alpha}\bar{\beta}\gamma} &= W_{\bar{\alpha}}(u_{\bar{\beta}\gamma}) - \Gamma^\mu_{\bar{\alpha}\bar{\beta}}u_{\mu\gamma} - \Gamma^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}}u_{\bar{\mu}\gamma} - \Gamma^\mu_{\bar{\alpha}\gamma}u_{\bar{\beta}\mu}.
\end{aligned}
\tag{3.4}
$$

Remark 3.1. (1) In (3.3) and (3.4), we have used $\Gamma^0_{ab} = 0$, $\Gamma^\gamma_{\alpha\bar{\beta}} = 0$ and $\Gamma^{\bar{\gamma}}_{\alpha\beta} = -\frac{i}{2}Q^{\bar{\gamma}}_{\beta\alpha}$ repeatedly.

(2) The complex conjugation can also be reflected in the indices of the components of any-order covariant derivative of a real function u, e.g. $\overline{u_{\alpha\beta}} = u_{\bar{\alpha}\bar{\beta}}$, $\overline{u_{\alpha\bar{\beta}\gamma}} = u_{\bar{\alpha}\beta\bar{\gamma}}$.

3.2. The sub-Laplacian.

On a contact Riemannian manifold M, with respect to a local $T^{(1,0)}M$-frame $\{W_j\}$, we define the sub-Laplacian operator as

$$\Delta_b u = u^\alpha{}_\alpha + u^{\bar{\alpha}}{}_{\bar{\alpha}},$$

for $u \in C_0^\infty(M)$. Furthermore, if $\{W_j\}$ is an orthonormal $T^{(1,0)}M$-frame, we have

$$\Delta_b u = u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha}. \tag{3.5}$$

For any functions $u \in C_0^\infty(M)$ and $v \in C^\infty(M)$, we define the L^2 inner product $(\cdot, \cdot)$ as

$$(u, v) = \int_M u\bar{v}dV. \tag{3.6}$$

For any vector field X, X^* is called the *formal adjoint* of X if $(Xu, v) = (u, X^*v)$ for $u, v \in C_0^\infty M$. And Δ_b is hypoelliptic and by a result of [21] has a discrete spectrum

$$0 < \lambda_1 < \lambda_2 < \cdots < \quad \uparrow +\infty.$$

Lemma 3.1. *We have*

$$W^*_\alpha = -W_{\bar{\alpha}} + \Gamma^\alpha_{\bar{\beta}\beta}, \qquad W^*_{\bar{\alpha}} = -W_\alpha + \Gamma^{\bar{\alpha}}_{\beta\bar{\beta}}, \qquad (iT)^* = iT. \tag{3.7}$$

Proof. By (2.14), we have

$$
\begin{aligned}
d\theta^n &= (-2i\theta^\alpha \wedge \theta^{\bar{\alpha}})^n = (-2)^n i^n n! \theta^1 \wedge \theta^{\bar{1}} \wedge \cdots \wedge \theta^n \wedge \theta^{\bar{n}} \\
&= (-2)^n i^n n! (-1)^{n(n-1)/2} \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n \wedge \theta^{\bar{1}} \wedge \cdots \wedge \theta^{\bar{n}} \\
&= (-2)^n i^{n^2} n! \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n \wedge \theta^{\bar{1}} \wedge \cdots \wedge \theta^{\bar{n}}.
\end{aligned}
$$

So the volume form is

$$dV := \theta \wedge d\theta^n = (-2)^n i^{n^2} n! \theta \wedge \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n \wedge \theta^{\bar{1}} \wedge \cdots \wedge \theta^{\bar{n}}. \tag{3.8}$$

For any vector field X and $u \in C_0^\infty(M)$, we get

$$\int_M XuvdV = \int_M vdu \wedge i_X dV = -\int_M udv \wedge i_X dV - \int_M uvd(i_X dV) + \int_M d(uvi_X dV)$$

$$= -\int_M uXvdV - \int_M uvd(i_X dV), \tag{3.9}$$

by Stokes' formula and $0 = \int_M i_X(vdu \wedge dV) = \int_M vXudV - \int_M vdu \wedge i_X dV$. It follows from the structure equation (2.14) that

$$d\theta^\beta = \theta^\gamma \wedge \omega_\gamma^\beta + \theta^{\bar\gamma} \wedge \omega_{\bar\gamma}^\beta = \Gamma_{\mu\gamma}^\beta \theta^\gamma \wedge \theta^\mu + \Gamma_{\bar\mu\gamma}^\beta \theta^\gamma \wedge \theta^{\bar\mu} + \Gamma_{\mu\bar\gamma}^\beta \theta^{\bar\gamma} \wedge \theta^\mu + \Gamma_{\bar\mu\bar\gamma}^\beta \theta^{\bar\gamma} \wedge \theta^{\bar\mu}, \quad mod \quad \theta. \tag{3.10}$$

Applying (3.8) and (3.10), we get

$$d(i_{W_\alpha} dV) = (-2)^n i^{n^2} n! d\left((-1)^\alpha \theta \wedge \theta^1 \wedge \cdots \wedge \widehat{\theta^\alpha} \wedge \cdots \wedge \theta^n \wedge \cdots \wedge \theta^{\bar n}\right)$$

$$= (-1)^\alpha (-2)^n i^{n^2} n!\left(\sum_{\beta<\alpha} (-1)^\beta \theta \wedge \theta^1 \wedge \cdots \wedge d\theta^\beta \wedge \cdots \wedge \widehat{\theta^\alpha} \wedge \cdots \wedge \theta^n \wedge \cdots \wedge \theta^{\bar n}\right.$$

$$+ \sum_{\beta>\alpha} (-1)^{\beta-1} \theta \wedge \theta^1 \wedge \cdots \wedge \widehat{\theta^\alpha} \cdots \wedge d\theta^\beta \wedge \cdots \wedge \theta^n \wedge \cdots \wedge \theta^{\bar n}$$

$$\left. + \sum_{\beta=1}^n (-1)^{n+\beta-1} \theta \wedge \theta^1 \wedge \cdots \wedge \widehat{\theta^\alpha} \cdots \wedge \theta^n \wedge \cdots \wedge d\theta^{\bar\beta} \wedge \cdots \wedge \theta^{\bar n}\right)$$

$$= (-2)^n i^{n^2} n!\left(-\Gamma_{\alpha\beta}^\beta + \Gamma_{\beta\alpha}^\beta - \Gamma_{\alpha\bar\beta}^{\bar\beta} + \Gamma_{\bar\beta\alpha}^{\bar\beta}\right) \theta \wedge \theta^1 \wedge \cdots \wedge \theta^{\bar n}$$

$$= \left(-\Gamma_{\alpha\beta}^\beta + \Gamma_{\beta\alpha}^\beta - \Gamma_{\alpha\bar\beta}^{\bar\beta} + \Gamma_{\bar\beta\alpha}^{\bar\beta}\right) dV = \Gamma_{\beta\alpha}^\beta dV.$$

The last identity holds because $\Gamma_{\alpha\beta}^\beta + \Gamma_{\alpha\bar\beta}^{\bar\beta} = 0$ by (2.11) and $\Gamma_{\bar\beta\alpha}^{\bar\beta} = 0$ by (2.7). For any $u \in C_0^\infty(M)$ and $v \in C^\infty(M)$, apply (3.9) with $X = W_\alpha$ to get

$$(W_\alpha u, v) = \int_M W_\alpha u\bar v dV = -\int_M uW_\alpha \bar v dV - \int_M u\bar v d(i_{W_\alpha} dV) = \int_M u\left(-W_\alpha \bar v - \Gamma_{\beta\alpha}^\beta \bar v\right) dV$$

$$= \int_M u\overline{\left(-W_{\bar\alpha} v + \Gamma_{\bar\beta\beta}^\alpha v\right)} dV = \left(u, \left(-W_{\bar\alpha} + \Gamma_{\bar\beta\beta}^\alpha\right)v\right), \tag{3.11}$$

by $\Gamma_{\beta\alpha}^\beta = -\Gamma_{\bar\beta\bar\beta}^{\bar\alpha}$ in (2.11). The first identity in (3.7) holds. The second identity in (3.7) follows from taking conjugation.

By the structure equation (2.14), $d\theta^\alpha$ doesn't contain $\theta \wedge \theta^\alpha$ terms and $d\theta^{\bar\alpha}$ doesn't contain $\theta \wedge \theta^{\bar\alpha}$ terms. So

$$d(i_T dV) = (-2)^n i^{n^2} n! d\left(\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n \wedge \theta^{\bar 1} \wedge \cdots \wedge \theta^{\bar n}\right) = 0. \tag{3.12}$$

Apply (3.9) with $X = T$ to get

$$(iTu, v) = i\int_M Tu\bar v dV = -i\int_M uT\bar v dV = (u, iTv).$$

$(iT)^* = iT$ follows. $\square$

Corollary 3.1. *With respect to an orthonormal $T^{(1,0)}M$-frame $\{W_j\}$, we have*

$$(\Delta_b u, v) = -\sum_\alpha (u_\alpha, v_\alpha) + (u_{\bar\alpha}, v_{\bar\alpha}),$$

for $u \in C_0^\infty(M)$ and $v \in C^\infty(M)$

Proof. By the definition of Δ_b, we get

$$(\Delta_b u, v) = \sum_\alpha (u_{\bar\alpha\alpha} + u_{\alpha\bar\alpha}, v) = \sum_{\alpha,\beta} \left(W_{\bar\alpha} u_\alpha - \Gamma^\beta_{\bar\alpha\alpha} u_\beta, v \right) + \left(W_\alpha u_{\bar\alpha} - \Gamma^{\bar\beta}_{\alpha\bar\alpha} u_{\bar\beta}, v \right)$$

$$= \sum_{\alpha,\beta} \left(u_\alpha, W_{\bar\alpha}^* v \right) - \left(\Gamma^\beta_{\bar\alpha\alpha} u_\beta, v \right) + \left(u_{\bar\alpha}, W_\alpha^* v \right) - \left(\Gamma^{\bar\beta}_{\alpha\bar\alpha} u_{\bar\beta}, v \right)$$

$$= -\sum_{\alpha,\beta} (u_\alpha, v_\alpha) + \left(u_\alpha, \Gamma^{\bar\alpha}_{\beta\bar\beta} v \right) - \left(\Gamma^\beta_{\bar\alpha\alpha} u_\beta, v \right) - (u_{\bar\alpha}, v_{\bar\alpha}) + \left(u_{\bar\alpha}, \Gamma^\alpha_{\bar\beta\beta} v \right) - \left(\Gamma^{\bar\beta}_{\alpha\bar\alpha} u_{\bar\beta}, v \right)$$

$$= -\sum_\alpha (u_\alpha, v_\alpha) + (u_{\bar\alpha}, v_{\bar\alpha}).$$

$\square$

Corollary 3.2.

$$(\Delta_b u)_\alpha = u_{\alpha\beta\bar\beta} + u_{\alpha\bar\beta\beta}. \tag{3.13}$$

Proof. By (3.4) and Corollary 3.1, we get

$$u_{\alpha\beta\bar\beta} + u_{\alpha\bar\beta\beta} = W_\alpha(u_{\beta\bar\beta}) - \Gamma^\mu_{\alpha\beta} u_{\mu\bar\beta} - \Gamma^{\bar\mu}_{\alpha\bar\beta} u_{\bar\mu\bar\beta} - \Gamma^{\bar\mu}_{\alpha\bar\beta} u_{\beta\bar\mu} + W_\alpha(u_{\bar\beta\beta}) - \Gamma^{\bar\mu}_{\alpha\bar\beta} u_{\bar\mu\beta} - \Gamma^\mu_{\alpha\beta} u_{\bar\beta\mu} - \Gamma^{\bar\mu}_{\alpha\beta} u_{\bar\beta\bar\mu}$$

$$= W_\alpha(u_{\beta\bar\beta} + u_{\bar\beta\beta}) - \left(\Gamma^\mu_{\alpha\beta} u_{\mu\bar\beta} + \Gamma^{\bar\mu}_{\alpha\bar\beta} u_{\beta\bar\mu} \right) - \left(\Gamma^{\bar\mu}_{\alpha\bar\beta} u_{\bar\mu\beta} + \Gamma^\mu_{\alpha\beta} u_{\bar\beta\mu} \right) - \left(\Gamma^{\bar\mu}_{\alpha\bar\beta} u_{\bar\mu\bar\beta} + \Gamma^{\bar\mu}_{\alpha\beta} u_{\bar\beta\bar\mu} \right)$$

$$= W_\alpha(u_{\beta\bar\beta} + u_{\bar\beta\beta}) = (\Delta_b u)_\alpha,$$

by the first two identities in (2.11). $\square$

3.3. The commutation formulae.

Proposition 3.1. *For the second-order covariant derivatives of a function u, we have the following commutation formulae.*

$$u_{ij} - u_{ji} = -\tau(W_i, W_j)u.$$

Proof. By the definition (3.1), we get $u_{ij} - u_{ji} = W_i W_j u - \nabla_{W_i} W_j u - W_j W_i u + \nabla_{W_j} W_i u = -(\nabla_{W_i} W_j - \nabla_{W_j} W_i - [W_i, W_j])u = -\tau(W_i, W_j)u.$ $\square$

In particular, Proposition 3.1 and (2.1) implies that

$$u_{\alpha\bar\beta} - u_{\bar\beta\alpha} = -\tau(W_\alpha, W_{\bar\beta})u = -2d\theta(W_\alpha, W_{\bar\beta})Tu = 2ih_{\alpha\bar\beta} u_0,$$

$$u_{\alpha\beta} = u_{\beta\alpha}, \tag{3.14}$$

$$u_{0\alpha} - u_{\alpha 0} = -\tau(T, W_\alpha) = -A_\alpha^{\bar\beta} u_{\bar\beta}.$$

Following Proposition 9.2 and 9.3 in [4] in the pseudohermitian case, we call the relations between the third-order covariant derivatives of functions u_{abc} and u_{acb} *the inner commutation formulae* and the relations between u_{abc} and u_{bac} *the outer commutation formulae*.

Proposition 3.2. *(1) We have the inner commutation formulae*

$$u_{\bar{\alpha}\beta\gamma} = u_{\bar{\alpha}\gamma\beta},$$

$$u_{\alpha\bar{\beta}\gamma} = u_{\alpha\gamma\bar{\beta}} - 2ih_{\gamma\bar{\beta}}u_{\alpha 0}. \tag{3.15}$$

(2) We have the outer commutation formulae

$$u_{\bar{\rho}\gamma\alpha} = u_{\gamma\bar{\rho}\alpha} - 2ih_{\gamma\bar{\rho}}u_{0\alpha} + u_{\beta}R_{\alpha}{}^{\beta}{}_{\gamma\bar{\rho}} + \frac{i}{2}Q_{\alpha\gamma,\bar{\rho}}^{\bar{\beta}}u_{\bar{\beta}},$$

$$u_{\bar{\rho}\bar{\gamma}\alpha} = u_{\bar{\gamma}\bar{\rho}\alpha} + 2i\left(A_{\bar{\gamma}}^{\beta}h_{\alpha\bar{\rho}} - A_{\bar{\rho}}^{\beta}h_{\alpha\bar{\gamma}}\right)u_{\beta} - \frac{i}{2}Q_{\bar{\rho}\bar{\beta},\alpha}^{\gamma}u_{\beta} - \frac{1}{4}Q_{\alpha\beta}^{\bar{\mu}}Q_{\bar{\rho}\bar{\beta}}^{\gamma}u_{\bar{\mu}}. \tag{3.16}$$

Proof. (1) The first identity of (3.15) follows directly from the second identity in (3.4) and $u_{\alpha\beta} = u_{\beta\alpha}$ in (3.14).

Taking conjugation of the fourth identity in (3.4), we get

$$u_{\alpha\gamma\bar{\beta}} = W_{\alpha}(u_{\gamma\bar{\beta}}) - \Gamma_{\alpha\gamma}^{\mu}u_{\mu\bar{\beta}} - \Gamma_{\alpha\gamma}^{\bar{\mu}}u_{\bar{\mu}\bar{\beta}} - \Gamma_{\alpha\bar{\beta}}^{\bar{\mu}}u_{\gamma\bar{\mu}}.$$

So by (2.11), (3.4) and (3.14), we get

$$u_{\alpha\bar{\beta}\gamma} = W_{\alpha}(u_{\bar{\beta}\gamma}) - \Gamma_{\alpha\bar{\beta}}^{\bar{\mu}}u_{\bar{\mu}\gamma} - \Gamma_{\alpha\gamma}^{\mu}u_{\bar{\beta}\mu} - \Gamma_{\alpha\gamma}^{\bar{\mu}}u_{\bar{\beta}\bar{\mu}}$$

$$= W_{\alpha}\left(u_{\gamma\bar{\beta}} - 2ih_{\gamma\bar{\beta}}u_0\right) - \Gamma_{\alpha\bar{\beta}}^{\bar{\mu}}\left(u_{\gamma\bar{\mu}} - 2ih_{\gamma\bar{\mu}}u_0\right) - \Gamma_{\alpha\gamma}^{\mu}\left(u_{\mu\bar{\beta}} - 2ih_{\mu\bar{\beta}}u_0\right) - \Gamma_{\alpha\gamma}^{\bar{\mu}}u_{\bar{\mu}\bar{\beta}}$$

$$= W_{\alpha}(u_{\gamma\bar{\beta}}) - \Gamma_{\alpha\gamma}^{\mu}u_{\mu\bar{\beta}} - \Gamma_{\alpha\gamma}^{\bar{\mu}}u_{\bar{\mu}\bar{\beta}} - \Gamma_{\alpha\bar{\beta}}^{\bar{\mu}}u_{\gamma\bar{\mu}} - 2i\left(h_{\gamma\bar{\beta}}W_{\alpha}(u_0) - \Gamma_{\alpha\bar{\beta}}^{\bar{\gamma}}u_0 - \Gamma_{\alpha\gamma}^{\beta}u_0\right)$$

$$= u_{\alpha\gamma\bar{\beta}} - 2ih_{\gamma\bar{\beta}}u_{\alpha 0}. \tag{3.17}$$

The second identity of (3.15) is proved.

(2) For the first identity in (3.16), note that

$$du_{\alpha} = (W_{\beta}u_{\alpha})\theta^{\beta} + (W_{\bar{\beta}}u_{\alpha})\theta^{\bar{\beta}} + T(u_{\alpha})\theta$$

$$= \left(u_{\beta\alpha} + \Gamma_{\beta\alpha}^{\rho}u_{\rho} - \frac{i}{2}Q_{\alpha\beta}^{\bar{\rho}}u_{\bar{\rho}}\right)\theta^{\beta} + (u_{\bar{\beta}\alpha} + \Gamma_{\bar{\beta}\alpha}^{\rho}u_{\rho})\theta^{\bar{\beta}} + (u_{0\alpha} + \Gamma_{0\alpha}^{\rho}u_{\rho})\theta \tag{3.18}$$

$$= u_{\beta\alpha}\theta^{\beta} + u_{\bar{\beta}\alpha}\theta^{\bar{\beta}} + u_{0\alpha}\theta + u_{\beta}\omega_{\alpha}^{\beta} - \frac{i}{2}Q_{\alpha\beta}^{\bar{\rho}}u_{\bar{\rho}}\theta^{\beta}$$

by using (3.3) and $\omega_{\alpha}^{\rho} = \Gamma_{\beta\alpha}^{\rho}\theta^{\beta} + \Gamma_{\bar{\beta}\alpha}^{\rho}\theta^{\bar{\beta}} + \Gamma_{0\alpha}^{\rho}\theta$. Taking exterior differentiation on both sides of (3.18), we get

$$0 = du_{\beta\alpha} \wedge \theta^{\beta} + du_{\bar{\beta}\alpha} \wedge \theta^{\bar{\beta}} + du_{0\alpha} \wedge \theta + u_{\beta\alpha}d\theta^{\beta} + u_{\bar{\beta}\alpha}d\theta^{\bar{\beta}} + u_{0\alpha}d\theta$$

$$+ du_{\beta} \wedge \omega_{\alpha}^{\beta} + u_{\beta}d\omega_{\alpha}^{\beta} - \frac{i}{2}d(Q_{\alpha\beta}^{\bar{\rho}}u_{\bar{\rho}})\wedge\theta^{\beta} - \frac{i}{2}Q_{\alpha\beta}^{\bar{\rho}}u_{\bar{\rho}}d\theta^{\beta}. \tag{3.19}$$

Note that

$$du_{\bar{\beta}\alpha} = W_\gamma(u_{\bar{\beta}\alpha})\theta^\gamma + W_{\bar{\gamma}}(u_{\bar{\beta}\alpha})\theta^{\bar{\gamma}} + T(u_{\bar{\beta}\alpha})\theta,$$

$$\omega_\alpha^\beta = \Gamma_{\gamma\alpha}^\beta \theta^\gamma + \Gamma_{\bar{\gamma}\alpha}^\beta \theta^{\bar{\gamma}}, \quad mod \quad \theta,$$

$$d\theta^\beta = \theta^\gamma \wedge \omega_\gamma^\beta + \theta^{\bar{\gamma}} \wedge \omega_{\bar{\gamma}}^\beta = \Gamma_{a\gamma}^\beta \theta^\gamma \wedge \theta^a + \Gamma_{a\bar{\gamma}}^\beta \theta^{\bar{\gamma}} \wedge \theta^a = \Gamma_{\bar{\rho}\gamma}^\beta \theta^\gamma \wedge \theta^{\bar{\rho}} + \Gamma_{\bar{\rho}\bar{\gamma}}^\beta \theta^{\bar{\gamma}} \wedge \theta^{\bar{\rho}}, \quad mod \quad \theta, \ \theta^\gamma \wedge \theta^\rho,$$

$$d\theta^{\bar{\beta}} = \theta^\gamma \wedge \omega_\gamma^{\bar{\beta}} + \theta^{\bar{\gamma}} \wedge \omega_{\bar{\gamma}}^{\bar{\beta}} = \Gamma_{a\gamma}^{\bar{\beta}} \theta^\gamma \wedge \theta^a + \Gamma_{a\bar{\gamma}}^{\bar{\beta}} \theta^{\bar{\gamma}} \wedge \theta^a = -\Gamma_{\gamma\rho}^{\bar{\beta}}\theta^\gamma \wedge \theta^\rho + \Gamma_{\bar{\rho}\gamma}^{\bar{\beta}} \theta^{\bar{\gamma}} \wedge \theta^\rho, \ mod \quad \theta, \ \theta^\gamma \wedge \theta^\rho,$$

$$d\omega_\alpha^\beta = \omega_\alpha^\mu \wedge \omega_\mu^\beta + \omega_\alpha^{\bar{\mu}} \wedge \omega_{\bar{\mu}}^\beta + R_{\alpha\ \lambda\bar{\mu}}^{\ \beta}\theta^\lambda \wedge \theta^{\bar{\mu}} + \frac{1}{2}R_{\alpha\ \bar{\lambda}\bar{\mu}}^{\ \beta}\theta^{\bar{\lambda}} \wedge \theta^{\bar{\mu}}$$

$$= \Gamma_{\gamma\alpha}^\mu \Gamma_{\bar{\rho}\mu}^\beta \theta^\gamma \wedge \theta^{\bar{\rho}} + \Gamma_{\bar{\gamma}\alpha}^\mu \Gamma_{\rho\mu}^\beta \theta^{\bar{\gamma}} \wedge \theta^\rho + \Gamma_{\gamma\alpha}^{\bar{\mu}} \Gamma_{\bar{\rho}\bar{\mu}}^\beta \theta^\gamma \wedge \theta^{\bar{\rho}} + R_{\alpha\ \gamma\bar{\rho}}^{\ \beta}\theta^\gamma \wedge \theta^{\bar{\rho}}$$

$$+ \Gamma_{\bar{\gamma}\alpha}^\mu \Gamma_{\bar{\rho}\mu}^\beta \theta^{\bar{\gamma}} \wedge \theta^{\bar{\rho}} + \frac{1}{2}R_{\alpha\ \bar{\gamma}\bar{\rho}}^{\ \beta}\theta^{\bar{\gamma}} \wedge \theta^{\bar{\rho}}, \hspace{4cm} mod \quad \theta, \ \theta^\gamma \wedge \theta^\rho,$$

$$(3.20)$$

by (2.14) and $\Gamma_{\alpha\bar{\beta}}^\gamma = 0$ in (2.7). Substituting (3.20) to the corresponding terms in (3.19) and comparing the coefficients of $\theta^\gamma \wedge \theta^{\bar{\rho}}$, we get

$$0 = -W_{\bar{\rho}}u_{\gamma\alpha} + W_\gamma u_{\bar{\rho}\alpha} + u_{\bar{\beta}\alpha}\Gamma_{\bar{\rho}\gamma}^\beta - u_{\bar{\beta}\alpha}\Gamma_{\gamma\bar{\rho}}^{\bar{\beta}} - 2ih_{\gamma\bar{\rho}}u_{0\alpha} + W_\gamma(u_\beta)\Gamma_{\bar{\rho}\alpha}^\beta - W_{\bar{\rho}}(u_\beta)\Gamma_{\gamma\alpha}^\beta$$

$$+ u_\beta\left(R_{\alpha\ \gamma\bar{\rho}}^{\ \beta} + \Gamma_{\gamma\alpha}^\mu\Gamma_{\bar{\rho}\mu}^\beta - \Gamma_{\bar{\rho}\alpha}^\mu\Gamma_{\gamma\mu}^\beta + \Gamma_{\gamma\alpha}^{\bar{\mu}}\Gamma_{\bar{\rho}\bar{\mu}}^\beta\right) + \frac{i}{2}W_{\bar{\rho}}\left(Q_{\alpha\gamma}^{\bar{\beta}}u_{\bar{\beta}}\right) - \frac{i}{2}Q_{\alpha\beta}^{\bar{\mu}}u_{\bar{\mu}}\Gamma_{\bar{\rho}\gamma}^\beta$$

$$= -W_{\bar{\rho}}u_{\gamma\alpha} + W_\gamma u_{\bar{\rho}\alpha} + u_{\bar{\beta}\alpha}\Gamma_{\bar{\rho}\gamma}^\beta - u_{\bar{\beta}\alpha}\Gamma_{\gamma\bar{\rho}}^{\bar{\beta}} - 2ih_{\gamma\bar{\rho}}u_{0\alpha} + u_\beta R_{\alpha\ \gamma\bar{\rho}}^{\ \beta}$$

$$+ \left(W_\gamma(u_\beta)\Gamma_{\bar{\rho}\alpha}^\beta - u_\mu\Gamma_{\gamma\beta}^\mu\Gamma_{\bar{\rho}\alpha}^\beta + \frac{i}{2}u_{\bar{\mu}}Q_{\beta\gamma}^{\bar{\mu}}\Gamma_{\bar{\rho}\alpha}^\beta\right) - \frac{i}{2}u_{\bar{\mu}}Q_{\beta\gamma}^{\bar{\mu}}\Gamma_{\bar{\rho}\alpha}^\beta - \left(W_{\bar{\rho}}(u_\beta)\Gamma_{\gamma\alpha}^\beta - u_\mu\Gamma_{\bar{\rho}\beta}^\mu\Gamma_{\gamma\alpha}^\beta\right)$$

$$- \frac{i}{2}u_\beta Q_{\alpha\gamma}^{\bar{\mu}}\Gamma_{\bar{\rho}\bar{\mu}}^\beta + \frac{i}{2}W_{\bar{\rho}}\left(Q_{\alpha\gamma}^{\bar{\beta}}u_{\bar{\beta}}\right) - \frac{i}{2}Q_{\alpha\beta}^{\bar{\mu}}u_{\bar{\mu}}\Gamma_{\bar{\rho}\gamma}^\beta. \hspace{3cm} (3.21)$$

Substitute

$$W_\gamma(u_\beta) - \Gamma_{\gamma\beta}^\mu u_\mu + \frac{i}{2}Q_{\beta\gamma}^{\bar{\mu}}u_{\bar{\mu}} = u_{\gamma\beta}, \qquad W_{\bar{\rho}}(u_\beta) - \Gamma_{\bar{\rho}\beta}^\mu u_\mu = u_{\bar{\rho}\beta},$$

by (3.3), into two brackets in (3.21) to get

$$0 = -W_{\bar{\rho}}u_{\gamma\alpha} + W_\gamma u_{\bar{\rho}\alpha} + u_{\bar{\beta}\alpha}\Gamma_{\bar{\rho}\gamma}^\beta - u_{\bar{\beta}\alpha}\Gamma_{\gamma\bar{\rho}}^{\bar{\beta}} - 2ih_{\gamma\bar{\rho}}u_{0\alpha} + u_\beta R_{\alpha\ \gamma\bar{\rho}}^{\ \beta} + u_{\gamma\beta}\Gamma_{\bar{\rho}\alpha}^\beta - u_{\bar{\rho}\beta}\Gamma_{\gamma\alpha}^\beta$$

$$- \frac{i}{2}u_\beta Q_{\alpha\gamma}^{\bar{\mu}}\Gamma_{\bar{\rho}\bar{\mu}}^\beta - \frac{i}{2}Q_{\beta\gamma}^{\bar{\mu}}u_{\bar{\mu}}\Gamma_{\bar{\rho}\alpha}^\beta + \frac{i}{2}W_{\bar{\rho}}(Q_{\alpha\gamma}^{\bar{\beta}})u_{\bar{\beta}} + \frac{i}{2}Q_{\alpha\gamma}^{\bar{\beta}}(W_{\bar{\rho}}u_{\bar{\beta}}) - \frac{i}{2}Q_{\alpha\beta}^{\bar{\mu}}u_{\bar{\mu}}\Gamma_{\bar{\rho}\gamma}^\beta$$

$$= \left(-W_{\bar{\rho}}u_{\gamma\alpha} + \Gamma_{\bar{\rho}\gamma}^\beta u_{\beta\alpha} + \Gamma_{\bar{\rho}\alpha}^\beta u_{\gamma\beta}\right) + \left(W_\gamma u_{\bar{\rho}\alpha} - u_{\bar{\beta}\alpha}\Gamma_{\gamma\bar{\rho}}^{\bar{\beta}} - u_{\bar{\rho}\beta}\Gamma_{\gamma\alpha}^\beta + \frac{i}{2}Q_{\alpha\gamma}^{\bar{\mu}}u_{\bar{\rho}\bar{\mu}}\right)$$

$$- \frac{i}{2}Q_{\alpha\gamma}^{\bar{\mu}}u_{\bar{\rho}\bar{\mu}} - 2ih_{\gamma\bar{\rho}}u_{0\alpha} + u_\beta R_{\alpha\ \gamma\bar{\rho}}^{\ \beta} + \frac{i}{2}Q_{\alpha\gamma}^{\bar{\mu}}\left(W_{\bar{\rho}}u_{\bar{\mu}} - \Gamma_{\bar{\rho}\bar{\mu}}^\beta u_\beta - \Gamma_{\bar{\rho}\bar{\mu}}^{\bar{\beta}}u_{\bar{\beta}}\right)$$

$$+ \frac{i}{2}\left(W_{\bar{\rho}}Q_{\alpha\gamma}^{\bar{\mu}} - \Gamma_{\bar{\rho}\alpha}^\beta Q_{\beta\gamma}^{\bar{\mu}} - \Gamma_{\bar{\rho}\gamma}^\beta Q_{\alpha\beta}^{\bar{\mu}} + \Gamma_{\bar{\rho}\bar{\beta}}^{\bar{\mu}}Q_{\alpha\gamma}^{\bar{\beta}}\right)u_{\bar{\mu}}$$

$$= -u_{\bar{\rho}\gamma\alpha} + u_{\gamma\bar{\rho}\alpha} - 2ih_{\gamma\bar{\rho}}u_{0\alpha} + u_\beta R_{\alpha\ \gamma\bar{\rho}}^{\ \beta} + \frac{i}{2}Q_{\alpha\gamma,\bar{\rho}}^{\bar{\mu}}u_{\bar{\mu}},$$

by (2.10), (3.3) and (3.4). The first identity of (3.16) holds.

To prove the second identity in (3.16), we consider the components of $\theta^{\bar\gamma} \wedge \theta^{\bar\rho}$ in (3.19) to get

$$0 = \left(W_{\bar\gamma}u_{\bar\rho\alpha} + u_{\beta\alpha}\Gamma^\beta_{\bar\rho\bar\gamma} + u_{\bar\beta\alpha}\Gamma^{\bar\beta}_{\bar\rho\bar\gamma} + u_\beta\Gamma^\mu_{\bar\gamma\alpha}\Gamma^\beta_{\bar\rho\mu} + \frac{1}{2}u_\beta R_\alpha{}^\beta{}_{\bar\gamma\bar\rho} + W_{\bar\gamma}u_\beta\Gamma^\beta_{\bar\rho\alpha} - \frac{i}{2}u_{\bar\mu}Q^{\bar\mu}_{\alpha\beta}\Gamma^\beta_{\bar\rho\bar\gamma}\right)\theta^{\bar\gamma} \wedge \theta^{\bar\rho}$$

$$= \left(W_{\bar\gamma}u_{\bar\rho\alpha} - u_{\beta\alpha}\Gamma^\beta_{\bar\gamma\bar\rho} - u_{\bar\beta\alpha}\Gamma^{\bar\beta}_{\bar\gamma\bar\rho} - u_{\bar\rho\beta}\Gamma^\beta_{\bar\gamma\alpha} + \frac{1}{2}u_\beta R_\alpha{}^\beta{}_{\bar\gamma\bar\rho} - \frac{i}{2}u_{\bar\mu}Q^{\bar\mu}_{\alpha\beta}\Gamma^\beta_{\bar\rho\bar\gamma}\right)\theta^{\bar\gamma} \wedge \theta^{\bar\rho}$$

$$= \left(u_{\bar\gamma\bar\rho\alpha} + \frac{1}{2}u_\beta R_\alpha{}^\beta{}_{\bar\gamma\bar\rho} + \frac{1}{4}u_{\bar\mu}Q^{\bar\mu}_{\alpha\beta}Q^\beta_{\bar\gamma\bar\rho}\right)\theta^{\bar\gamma} \wedge \theta^{\bar\rho},$$

by

$$\left(W_{\bar\gamma}u_\beta\Gamma^\beta_{\bar\rho\alpha} + u_\beta\Gamma^\mu_{\bar\gamma\alpha}\Gamma^\beta_{\bar\rho\mu}\right)\theta^{\bar\gamma} \wedge \theta^{\bar\rho} = \left(-W_{\bar\rho}u_\beta\Gamma^\beta_{\bar\gamma\alpha} + u_\mu\Gamma^\mu_{\bar\rho\beta}\Gamma^\beta_{\bar\gamma\alpha}\right)\theta^{\bar\gamma} \wedge \theta^{\bar\rho} = -u_{\bar\rho\beta}\Gamma^\beta_{\bar\gamma\alpha}\theta^{\bar\gamma} \wedge \theta^{\bar\rho},$$

by using (3.3). Equivalently, we have

$$0 = u_{\bar\gamma\bar\rho\alpha} - u_{\bar\rho\bar\gamma\alpha} + \frac{1}{2}u_\beta R_\alpha{}^\beta{}_{\bar\gamma\bar\rho} - \frac{1}{2}u_\beta R_\alpha{}^\beta{}_{\bar\rho\bar\gamma} + \frac{1}{4}u_{\bar\mu}Q^{\bar\mu}_{\alpha\beta}\left(Q^\beta_{\bar\gamma\bar\rho} - Q^\beta_{\bar\rho\bar\gamma}\right)$$

$$= u_{\bar\gamma\bar\rho\alpha} - u_{\bar\rho\bar\gamma\alpha} + u_\beta R_\alpha{}^\beta{}_{\bar\gamma\bar\rho} + \frac{1}{4}Q^{\bar\mu}_{\alpha\beta}Q^\rho_{\bar\gamma\bar\beta}u_{\bar\mu} = u_{\bar\gamma\bar\rho\alpha} - u_{\bar\rho\bar\gamma\alpha} + u_\beta R_\alpha{}^\beta{}_{\bar\gamma\bar\rho} - \frac{1}{4}Q^{\bar\mu}_{\alpha\beta}Q^\gamma_{\bar\rho\bar\beta}u_{\bar\mu},$$

by (2.11). This together with $R_\alpha{}^\beta{}_{\bar\gamma\bar\rho} = 2i(A^\beta_{\bar\gamma}h_{\alpha\bar\rho} - A^\beta_{\bar\rho}h_{\alpha\bar\gamma}) - \frac{i}{2}h^{\beta\bar\sigma}h_{\lambda\bar\gamma}Q^\lambda_{\bar\rho\bar\sigma,\alpha}$ (cf. (43) in [4]) implies the second identity in (3.16). □

Remark 3.2. Note that when $Q \equiv 0$, the commutative formulae (3.16) is the same as Proposition 9.2 in [4] in pseudohermitian case.

4. The Bochner-type formula

By definition, we have $(\nabla_H u)^\alpha = h(\nabla_H u, W_{\bar\alpha}) = W_{\bar\alpha}u = u_{\bar\alpha}$ for any $u \in C_0^\infty(M)$. Thus $\nabla_H u = (\nabla_H u)^\alpha W_\alpha + (\nabla_H u)^{\bar\alpha}W_{\bar\alpha} = u_{\bar\alpha}W_\alpha + u_\alpha W_{\bar\alpha}$. Consequently, we have $\partial_b u = u_{\bar\alpha}W_\alpha$ and $\|\partial_b u\|^2 = \|u_{\bar\lambda}W_\lambda\|^2 = u_\lambda u_{\bar\lambda}$.

Theorem 4.1. *Under an orthonormal $T^{(1,0)}M$-frame, the Bochner-type formula holds in the following form:*

$$\Delta_b(\|\partial_b u\|^2) = 2\left(u_{\alpha\lambda}u_{\bar\alpha\bar\lambda} + u_{\alpha\bar\lambda}u_{\bar\alpha\lambda}\right) + 4i(u_\alpha u_{0\bar\alpha} - u_{\bar\alpha}u_{0\alpha})$$

$$+ 2ni(A_{\bar\alpha\bar\beta}u_\alpha u_\beta - A_{\alpha\beta}u_{\bar\alpha}u_{\bar\beta}) + 2R_{\alpha\bar\beta}u_{\bar\alpha}u_\beta + u_\alpha(\Delta_b u)_{\bar\alpha} + u_{\bar\alpha}(\Delta_b u)_\alpha$$

$$+ i\left(Q^\gamma_{\bar\alpha\bar\beta,\gamma}u_\alpha u_\beta - Q^{\bar\gamma}_{\alpha\beta,\bar\gamma}u_{\bar\alpha}u_{\bar\beta}\right) - \frac{1}{2}Q^{\bar\rho}_{\alpha\gamma}Q^\rho_{\bar\beta\bar\gamma}u_{\bar\alpha}u_\beta. \tag{4.1}$$

To prove it, we need a lemma.

Lemma 4.1. *For any $u \in C_0^\infty(M)$,*

$$\Delta_b(\|\partial_b u\|^2) = S_1 + S_2, \tag{4.2}$$

where $S_1 = 2\left(u_{\alpha\lambda}u_{\bar\alpha\bar\lambda} + u_{\alpha\bar\lambda}u_{\bar\alpha\lambda}\right)$, $S_2 = u_{\bar\lambda}u_{\alpha\bar\alpha\lambda} + u_\lambda u_{\alpha\bar\alpha\bar\lambda} + u_\lambda u_{\bar\alpha\alpha\bar\lambda} + u_{\bar\lambda}u_{\bar\alpha\alpha\lambda}$.

Proof. We claim that

$$\left(\|\partial_b u\|^2\right)_{\alpha\bar\alpha} = u_{\alpha\lambda}u_{\bar\alpha\bar\lambda} + u_{\alpha\bar\lambda}u_{\bar\alpha\lambda} + u_{\bar\lambda}u_{\alpha\bar\alpha\lambda} + u_\lambda u_{\alpha\bar\alpha\bar\lambda}. \tag{4.3}$$

Then (4.2) follows directly by taking summation of (4.3) and its conjugation.

By (2.11), we have

$$\Gamma^\beta_{\alpha\lambda}u_\beta u_{\bar\lambda} + \Gamma^{\bar\beta}_{\alpha\bar\lambda}u_\lambda u_{\bar\beta} = -\Gamma^{\bar\lambda}_{\alpha\bar\beta}u_\beta u_{\bar\lambda} + \Gamma^{\bar\beta}_{\alpha\bar\lambda}u_\lambda u_{\bar\beta} = 0,$$

and $\Gamma^{\bar\beta}_{\alpha\lambda}u_{\bar\beta}u_{\bar\lambda} = -\Gamma^{\bar\lambda}_{\alpha\bar\beta}u_{\bar\beta}u_{\bar\lambda} = -\Gamma^{\bar\beta}_{\alpha\lambda}u_{\bar\beta}u_{\bar\lambda}$. Thus, $\Gamma^{\bar\beta}_{\alpha\lambda}u_{\bar\beta}u_{\bar\lambda} = 0$. So by (3.3), we get

$$W_\alpha\big(\|\partial_b u\|^2\big) = W_\alpha(u_\lambda)u_{\bar\lambda} + u_\lambda W_\alpha(u_{\bar\lambda})$$
$$= \Big(u_{\alpha\lambda} + \Gamma^\beta_{\alpha\lambda}u_\beta + \Gamma^{\bar\beta}_{\alpha\lambda}u_{\bar\beta}\Big)u_{\bar\lambda} + u_\lambda\Big(u_{\alpha\bar\lambda} + \Gamma^{\bar\beta}_{\alpha\bar\lambda}u_{\bar\beta}\Big) = u_{\alpha\lambda}u_{\bar\lambda} + u_\lambda u_{\alpha\bar\lambda}. \qquad (4.4)$$

Then taking conjugation of (4.4), we get $W_{\bar\alpha}\big(\|\partial_b u\|^2\big) = u_{\bar\alpha\bar\lambda}u_\lambda + u_{\bar\lambda}u_{\bar\alpha\lambda}$. So

$$\big(\|\partial_b u\|^2\big)_{\alpha\bar\alpha} = W_\alpha W_{\bar\alpha}(\|\partial_b u\|^2) - \Gamma^{\bar\gamma}_{\alpha\bar\alpha}W_{\bar\gamma}(\|\partial_b u\|^2)$$
$$= W_\alpha\big(u_{\bar\alpha\bar\lambda}u_\lambda + u_{\bar\lambda}u_{\bar\alpha\lambda}\big) - \Gamma^{\bar\gamma}_{\alpha\bar\alpha}\big(u_{\bar\gamma\bar\lambda}u_\lambda + u_{\bar\gamma\lambda}u_{\bar\lambda}\big)$$
$$= W_\alpha(u_\lambda)u_{\bar\alpha\bar\lambda} + W_\alpha(u_{\bar\lambda})u_{\bar\alpha\lambda} + \Big(W_\alpha(u_{\bar\alpha\bar\lambda}) - \Gamma^{\bar\gamma}_{\alpha\bar\alpha}u_{\bar\gamma\bar\lambda}\Big)u_\lambda + \Big(W_\alpha(u_{\bar\alpha\lambda}) - \Gamma^{\bar\gamma}_{\alpha\bar\alpha}u_{\bar\gamma\lambda}\Big)u_{\bar\lambda}$$
$$= \Big(u_{\alpha\lambda} + \Gamma^\gamma_{\alpha\lambda}u_\gamma + \Gamma^{\bar\gamma}_{\alpha\lambda}u_{\bar\gamma}\Big)u_{\bar\alpha\bar\lambda} + \Big(u_{\alpha\bar\lambda} + \Gamma^{\bar\gamma}_{\alpha\bar\lambda}u_{\bar\gamma}\Big)u_{\bar\alpha\lambda}$$
$$+ \Big(u_{\alpha\bar\alpha\bar\lambda} + \Gamma^{\bar\gamma}_{\alpha\bar\lambda}u_{\bar\alpha\bar\gamma}\Big)u_\lambda + \Big(u_{\alpha\bar\alpha\lambda} + \Gamma^\gamma_{\alpha\lambda}u_{\bar\alpha\gamma} + \Gamma^{\bar\gamma}_{\alpha\lambda}u_{\bar\alpha\bar\gamma}\Big)u_{\bar\lambda}$$
$$= u_{\alpha\lambda}u_{\bar\alpha\bar\lambda} + u_{\alpha\bar\lambda}u_{\bar\alpha\lambda} + u_\lambda u_{\alpha\bar\alpha\bar\lambda} + u_{\bar\lambda}u_{\alpha\bar\alpha\lambda}$$
$$+ \Big(\Gamma^\gamma_{\alpha\lambda}u_\gamma u_{\bar\alpha\bar\lambda} + \Gamma^{\bar\gamma}_{\alpha\bar\lambda}u_\lambda u_{\bar\alpha\bar\gamma}\Big) + \Big(\Gamma^{\bar\gamma}_{\alpha\lambda}u_{\bar\gamma}u_{\bar\alpha\bar\lambda} + \Gamma^{\bar\gamma}_{\alpha\lambda}u_{\bar\lambda}u_{\bar\alpha\bar\gamma}\Big) + \Big(\Gamma^{\bar\gamma}_{\alpha\bar\lambda}u_{\bar\gamma}u_{\bar\alpha\lambda} + \Gamma^\gamma_{\alpha\lambda}u_{\bar\lambda}u_{\bar\alpha\gamma}\Big),$$

where we have used (3.3) and (3.4). The result follows from

$$\Gamma^\gamma_{\alpha\lambda}u_\gamma u_{\bar\alpha\bar\lambda} + \Gamma^{\bar\gamma}_{\alpha\bar\lambda}u_\lambda u_{\bar\alpha\bar\gamma} = -\Gamma^{\bar\lambda}_{\alpha\bar\gamma}u_\gamma u_{\bar\alpha\bar\lambda} + \Gamma^{\bar\gamma}_{\alpha\bar\lambda}u_\lambda u_{\bar\alpha\bar\gamma} = 0,$$

and similarly, $\Gamma^{\bar\gamma}_{\alpha\lambda}u_{\bar\gamma}u_{\bar\alpha\bar\lambda} + \Gamma^{\bar\gamma}_{\alpha\lambda}u_{\bar\lambda}u_{\bar\alpha\bar\gamma} = 0$, $\Gamma^{\bar\gamma}_{\alpha\bar\lambda}u_{\bar\gamma}u_{\bar\alpha\lambda} + \Gamma^\gamma_{\alpha\lambda}u_{\bar\lambda}u_{\bar\alpha\gamma} = 0$ by (2.11). $\qquad\square$

(4.2) coincides with (9.34) in [10] in the pseudohermitian case.

Proof of theorem4.1. Note that by (3.13), we have $(\Delta_b u)_c = u_{c\alpha\bar\alpha} + u_{c\bar\alpha\alpha}$. We hope to express third-order covariant derivatives in (4.2) in terms of $(\Delta_b u)_c$. To do so, we apply inner and outer commutation formulae to (4.2) to express $u_{\alpha\bar\alpha b}$ and $u_{\bar\alpha\alpha b}$ in terms of $u_{b\alpha\bar\alpha}$ and $u_{b\bar\alpha\alpha}$, respectively. By (3.15), we have the following inner commutation formulae.

$$u_{\alpha\bar\alpha\lambda} = u_{\alpha\lambda\bar\alpha} - 2ih_{\lambda\bar\alpha}u_{\alpha 0}, \quad u_{\alpha\bar\alpha\bar\lambda} = u_{\alpha\bar\lambda\bar\alpha},$$
$$u_{\bar\alpha\alpha\bar\lambda} = u_{\bar\alpha\bar\lambda\alpha} + 2ih_{\alpha\bar\lambda}u_{\bar\alpha 0}, \quad u_{\bar\alpha\alpha\lambda} = u_{\bar\alpha\lambda\alpha}.$$

So S_2 in (4.2) becomes

$$S_2 = u_{\bar\lambda}(u_{\alpha\lambda\bar\alpha} - 2ih_{\lambda\bar\alpha}u_{\alpha 0}) + u_\lambda u_{\alpha\bar\lambda\bar\alpha} + u_\lambda(u_{\bar\alpha\bar\lambda\alpha} + 2ih_{\alpha\bar\lambda}u_{\bar\alpha 0}) + u_{\bar\lambda}u_{\bar\alpha\lambda\alpha}$$
$$= u_{\bar\lambda}u_{\alpha\lambda\bar\alpha} + u_\lambda u_{\alpha\bar\lambda\bar\alpha} + u_\lambda u_{\bar\alpha\bar\lambda\alpha} + u_{\bar\lambda}u_{\bar\alpha\lambda\alpha} - 2iu_{\bar\alpha}u_{0\alpha} + 2iu_\alpha u_{0\bar\alpha} - 2iA^{\bar\beta}_\alpha u_{\bar\alpha}u_{\bar\beta} + 2iA^\beta_\alpha u_\alpha u_\beta,$$
$$(4.5)$$

by $u_{\alpha 0} = u_{0\alpha} + A_{\alpha}^{\bar{\beta}} u_{\bar{\beta}}$, $u_{\bar{\alpha}0} = u_{0\bar{\alpha}} + A_{\bar{\alpha}}^{\beta} u_{\beta}$ in (3.14). The outer commutation formulae (3.16) for $\rho = \alpha$ can be written as

$$
\begin{aligned}
u_{\bar{\alpha}\lambda\alpha} &= u_{\lambda\bar{\alpha}\alpha} - 2iu_{0\lambda} + R_{\lambda\bar{\beta}}u_{\beta} - \frac{i}{2}Q_{\bar{\beta}\lambda,\bar{\alpha}}^{\bar{\alpha}}u_{\bar{\beta}}, \\
u_{\bar{\alpha}\bar{\lambda}\alpha} &= u_{\bar{\lambda}\bar{\alpha}\alpha} + 2i(n-1)A_{\bar{\lambda}}^{\beta}u_{\beta} + \frac{i}{2}Q_{\bar{\lambda}\beta,\alpha}^{\alpha}u_{\beta} - \frac{1}{4}Q_{\alpha\beta}^{\bar{\mu}}Q_{\bar{\alpha}\bar{\beta}}^{\lambda}u_{\bar{\mu}}.
\end{aligned}
\tag{4.6}
$$

by $Q_{\alpha\lambda,\bar{\alpha}}^{\bar{\beta}} = -Q_{\beta\lambda,\bar{\alpha}}^{\bar{\alpha}}$ in (2.11) and

$$
R_{\alpha\ \lambda\bar{\alpha}}^{\ \beta} = h^{\beta\bar{\mu}}R_{\alpha\bar{\mu}\lambda\bar{\alpha}} = R_{\alpha\bar{\beta}\lambda\bar{\alpha}} = R_{\bar{\beta}\alpha\bar{\alpha}\lambda} = R_{\bar{\alpha}\alpha\bar{\beta}\lambda} = R_{\alpha\bar{\alpha}\lambda\bar{\beta}} = R_{\lambda\bar{\alpha}\alpha\bar{\beta}} = R_{\lambda\ \alpha\bar{\beta}}^{\ \alpha} = R_{\lambda\bar{\beta}}, \tag{4.7}
$$

by using Proposition 2.6 repeatedly. Taking conjugation of (4.6) and noting that $R_{\bar{\lambda}\beta} = R_{\bar{\lambda}\ \bar{\alpha}\beta}^{\ \bar{\alpha}} = R_{\bar{\lambda}\alpha\bar{\alpha}\beta} = R_{\beta\bar{\lambda}}$ by (4.7), we get

$$
\begin{aligned}
u_{\alpha\bar{\lambda}\bar{\alpha}} &= u_{\bar{\lambda}\alpha\bar{\alpha}} + 2iu_{0\bar{\lambda}} + R_{\beta\bar{\lambda}}u_{\bar{\beta}} + \frac{i}{2}Q_{\beta\bar{\lambda},\alpha}^{\alpha}u_{\beta}, \\
u_{\alpha\lambda\bar{\alpha}} &= u_{\lambda\alpha\bar{\alpha}} - 2i(n-1)A_{\lambda}^{\bar{\beta}}u_{\bar{\beta}} - \frac{i}{2}Q_{\bar{\lambda}\beta,\bar{\alpha}}^{\bar{\alpha}}u_{\bar{\beta}} - \frac{1}{4}Q_{\alpha\beta}^{\bar{\lambda}}Q_{\bar{\alpha}\bar{\beta}}^{\mu}u_{\mu}.
\end{aligned}
\tag{4.8}
$$

Substitute (4.6) and (4.8) to (4.5) to get

$$
\begin{aligned}
S_2 &= u_{\bar{\lambda}}u_{\lambda\alpha\bar{\alpha}} - 2i(n-1)A_{\lambda}^{\bar{\beta}}u_{\bar{\lambda}}u_{\bar{\beta}} - \frac{i}{2}Q_{\bar{\lambda}\beta,\bar{\alpha}}^{\bar{\alpha}}u_{\bar{\lambda}}u_{\bar{\beta}} - \frac{1}{4}Q_{\alpha\beta}^{\bar{\lambda}}Q_{\bar{\alpha}\bar{\beta}}^{\mu}u_{\mu}u_{\bar{\lambda}} \\
&\quad + u_{\lambda}u_{\bar{\lambda}\alpha\bar{\alpha}} + 2iu_{\lambda}u_{0\bar{\lambda}} + R_{\beta\bar{\lambda}}u_{\lambda}u_{\bar{\beta}} + \frac{i}{2}Q_{\beta\bar{\lambda},\alpha}^{\alpha}u_{\lambda}u_{\beta} \\
&\quad + u_{\lambda}u_{\bar{\lambda}\bar{\alpha}\alpha} + 2i(n-1)A_{\bar{\lambda}}^{\beta}u_{\lambda}u_{\beta} + \frac{i}{2}Q_{\bar{\lambda}\beta,\alpha}^{\alpha}u_{\lambda}u_{\beta} - \frac{1}{4}Q_{\alpha\beta}^{\bar{\mu}}Q_{\bar{\alpha}\bar{\beta}}^{\lambda}u_{\lambda}u_{\bar{\mu}} \\
&\quad + u_{\bar{\lambda}}u_{\lambda\bar{\alpha}\alpha} - 2iu_{\bar{\lambda}}u_{0\lambda} + R_{\lambda\bar{\beta}}u_{\beta}u_{\bar{\lambda}} - \frac{i}{2}Q_{\bar{\beta}\lambda,\bar{\alpha}}^{\bar{\alpha}}u_{\bar{\beta}}u_{\bar{\lambda}} \\
&\quad - 2iu_{\bar{\alpha}}u_{0\alpha} + 2iu_{\alpha}u_{0\bar{\alpha}} - 2iA_{\alpha}^{\bar{\beta}}u_{\bar{\alpha}}u_{\bar{\beta}} + 2iA_{\bar{\alpha}}^{\beta}u_{\alpha}u_{\beta} \\
&= u_{\lambda}(\Delta_b u)_{\bar{\lambda}} + u_{\bar{\lambda}}(\Delta_b u)_{\lambda} + 4i(u_{\alpha}u_{0\bar{\alpha}} - u_{\bar{\alpha}}u_{0\alpha}) + 2ni(A_{\bar{\alpha}\bar{\beta}}u_{\alpha}u_{\beta} - A_{\alpha\beta}u_{\bar{\alpha}}u_{\bar{\beta}}) + 2R_{\alpha\bar{\beta}}u_{\bar{\alpha}}u_{\beta} \\
&\quad + iQ_{\lambda\bar{\beta},\alpha}^{\alpha}u_{\lambda}u_{\beta} - iQ_{\lambda\beta,\bar{\alpha}}^{\bar{\alpha}}u_{\bar{\lambda}}u_{\bar{\beta}} - \frac{1}{2}Q_{\mu\beta}^{\bar{\alpha}}Q_{\lambda\bar{\beta}}^{\alpha}u_{\lambda}u_{\bar{\mu}}.
\end{aligned}
\tag{4.9}
$$

The last identity follows from $u_{\lambda}u_{\bar{\lambda}\alpha\bar{\alpha}} + u_{\lambda}u_{\bar{\lambda}\bar{\alpha}\alpha} = u_{\lambda}(\Delta_b u)_{\bar{\lambda}}$ by (3.13), $A_{\alpha}^{\bar{\beta}} = h^{\gamma\bar{\beta}}A_{\alpha\gamma} = A_{\alpha\beta}$ and $Q_{\beta\alpha}^{\bar{\gamma}} = -Q_{\gamma\alpha}^{\bar{\beta}}$ in (2.11). Substituting (4.9) to (4.2), we get (4.1).

Remark 4.1. When $Q \equiv 0$, the Bochner-type formula (4.1) is the same as the pseudohermitian case (see (9.36) in [10] or Theorem 6 in [19]).

5. Two useful identities

We need the following lemma to handle the second bracket in the Bochner type formula (4.1).

Lemma 5.1. *For any $u \in C_0^{\infty}(M)$, we have*

$$
\begin{aligned}
\int_M i(u_{0\bar{\alpha}}u_{\alpha} - u_{0\alpha}u_{\bar{\alpha}})dV = &\frac{1}{n}\int_M \left(u_{\bar{\alpha}\beta}u_{\alpha\bar{\beta}} - u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}} - R_{\alpha\bar{\beta}}u_{\bar{\alpha}}u_{\beta} - \frac{i}{2}Q_{\bar{\alpha}\bar{\beta},\gamma}^{\gamma}u_{\alpha}u_{\beta} \right. \\
&\left. + \frac{i}{2}Q_{\alpha\beta,\bar{\gamma}}^{\bar{\gamma}}u_{\bar{\alpha}}u_{\bar{\beta}} - \frac{1}{2}Q_{\alpha\gamma}^{\bar{\rho}}Q_{\bar{\beta}\bar{\gamma}}^{\rho}u_{\bar{\alpha}}u_{\beta} + Q_{\alpha\gamma}^{\bar{\rho}}Q_{\bar{\beta}\bar{\rho}}^{\gamma}u_{\bar{\alpha}}u_{\beta} \right)dV,
\end{aligned}
\tag{5.1}
$$

and

$$\int_M i(u_{0\bar{\alpha}}u_\alpha - u_{0\alpha}u_{\bar{\alpha}})dV = \int_M \left(-\frac{2}{n}\left|\sum_\alpha u_{\alpha\alpha}\right|^2 + \frac{1}{2n}(\Delta_b u)^2 + iA_{\alpha\beta}u_{\bar{\alpha}}u_{\bar{\beta}} - iA_{\bar{\alpha}\bar{\beta}}u_\alpha u_\beta \right)dV. \quad (5.2)$$

Remark 5.1. (5.1) and (5.2) are the same as the corresponding identities in the pseudohermitian case when $Q \equiv 0$ (cf. Lemma 9.1 in [10] or Lemma 4 and Lemma 5 in [13]).

5.1. **The Proof of** (5.1). By definition we have

$$\int_M u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}}dV = \sum_{\alpha,\beta}(u_{\alpha\beta}, u_{\alpha\beta}) = \sum_{\alpha,\beta,\gamma}\left(u_{\alpha\beta}, W_\alpha(u_\beta) - \Gamma_{\alpha\beta}^\gamma u_\gamma - \Gamma_{\alpha\beta}^{\bar{\gamma}}u_{\bar{\gamma}}\right)$$

$$= \sum_{\alpha,\beta}\left(-W_{\bar{\alpha}}(u_{\alpha\beta}) + \Gamma_{\bar{\gamma}\bar{\alpha}}^\alpha u_{\alpha\beta} - \Gamma_{\bar{\alpha}\bar{\gamma}}^{\bar{\beta}}u_{\alpha\gamma}, u_\beta\right) - \left(\Gamma_{\bar{\alpha}\bar{\gamma}}^\beta u_{\alpha\gamma}, u_{\bar{\beta}}\right)$$

$$= \sum_{\alpha,\beta,\gamma}\left(-W_{\bar{\alpha}}\left(W_\alpha(u_\beta) - \Gamma_{\alpha\beta}^\gamma u_\gamma - \Gamma_{\alpha\beta}^{\bar{\gamma}}u_{\bar{\gamma}}\right) + \Gamma_{\bar{\gamma}\bar{\gamma}}^\alpha u_{\alpha\beta} - \Gamma_{\bar{\alpha}\bar{\gamma}}^{\bar{\beta}}u_{\alpha\gamma}, u_\beta\right) - \left(\Gamma_{\bar{\alpha}\bar{\gamma}}^\beta u_{\alpha\gamma}, u_{\bar{\beta}}\right)$$

$$\quad (5.3)$$

by (3.3) and Lemma 3.1. Apply $[W_{\bar{\alpha}}, W_\alpha] = \nabla_{W_{\bar{\alpha}}}W_\alpha - \nabla_{W_\alpha}W_{\bar{\alpha}} - \tau(W_{\bar{\alpha}}, W_\alpha) = \Gamma_{\bar{\alpha}\alpha}^\gamma W_\gamma - \Gamma_{\alpha\bar{\alpha}}^{\bar{\gamma}}W_{\bar{\gamma}} - 2h(W_{\bar{\alpha}}, JW_\alpha)T = \Gamma_{\bar{\alpha}\alpha}^\gamma W_\gamma - \Gamma_{\alpha\bar{\alpha}}^{\bar{\gamma}}W_{\bar{\gamma}} - 2i\delta_{\alpha\bar{\alpha}}T$ to (5.3) to get that $\int_M u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}}dV$ equals to

$$-\sum_{\alpha,\beta}\left(W_\alpha W_{\bar{\alpha}}(u_\beta), u_\beta\right) + 2ni\sum_\beta\left(T(u_\beta), u_\beta\right) + \sum_{\alpha,\beta,\gamma}\left(-\Gamma_{\bar{\alpha}\alpha}^\gamma W_\gamma(u_\beta) + \Gamma_{\alpha\bar{\alpha}}^{\bar{\gamma}}W_{\bar{\gamma}}(u_\beta)\right.$$

$$\left.+ W_{\bar{\alpha}}\left(\Gamma_{\alpha\beta}^\gamma u_\gamma + \Gamma_{\alpha\beta}^{\bar{\gamma}}u_{\bar{\gamma}}\right) + \Gamma_{\bar{\gamma}\bar{\gamma}}^\alpha u_{\alpha\beta} - \Gamma_{\bar{\alpha}\bar{\gamma}}^{\bar{\beta}}u_{\alpha\gamma}, u_\beta\right) - \sum_{\alpha,\beta,\gamma}\left(\Gamma_{\bar{\alpha}\bar{\gamma}}^\beta u_{\alpha\gamma}, u_{\bar{\beta}}\right)$$

$$= -\sum_{\alpha,\beta}\left(W_\alpha(u_{\bar{\alpha}\beta} + \Gamma_{\bar{\alpha}\beta}^\gamma u_\gamma), u_\beta\right) + 2ni\sum_\beta\left(T(u_\beta), u_\beta\right)$$

$$+ \sum_{\alpha,\beta,\gamma}\left(-\Gamma_{\bar{\alpha}\alpha}^\gamma W_\gamma(u_\beta) + \Gamma_{\alpha\bar{\alpha}}^{\bar{\gamma}}W_{\bar{\gamma}}(u_\beta) + W_{\bar{\alpha}}(\Gamma_{\alpha\beta}^\gamma)u_\gamma + \Gamma_{\alpha\beta}^\gamma W_{\bar{\alpha}}(u_\gamma) + W_{\bar{\alpha}}(\Gamma_{\alpha\beta}^{\bar{\gamma}})u_{\bar{\gamma}} + \Gamma_{\alpha\beta}^{\bar{\gamma}}W_{\bar{\alpha}}(u_{\bar{\gamma}})\right.$$

$$\left.+ \Gamma_{\bar{\gamma}\gamma}^\alpha u_{\alpha\beta} - \Gamma_{\bar{\alpha}\bar{\gamma}}^{\bar{\beta}}u_{\alpha\gamma}, u_\beta\right) - \sum_{\alpha,\beta,\gamma}\left(\Gamma_{\bar{\alpha}\bar{\gamma}}^\beta u_{\alpha\gamma}, u_{\bar{\beta}}\right)$$

$$= -\sum_{\alpha,\beta}\left(W_\alpha(u_{\bar{\alpha}\beta}), u_\beta\right) - \sum_{\alpha,\beta,\gamma}\left(W_\alpha(\Gamma_{\bar{\alpha}\beta}^\gamma)u_\gamma, u_\beta\right) - \sum_{\alpha,\beta,\gamma,\rho}\left(\Gamma_{\bar{\alpha}\beta}^\gamma u_{\alpha\gamma} + \Gamma_{\bar{\alpha}\beta}^\gamma \Gamma_{\alpha\gamma}^\rho u_\rho + \Gamma_{\bar{\alpha}\beta}^\gamma \Gamma_{\alpha\gamma}^{\bar{\rho}}u_{\bar{\rho}}, u_\beta\right)$$

$$+ 2ni\sum_\beta\left(T(u_\beta), u_\beta\right) + \sum_{\alpha,\beta,\gamma,\rho}\left(-\Gamma_{\bar{\alpha}\alpha}^\gamma(u_{\gamma\beta} + \Gamma_{\gamma\beta}^\rho u_\rho + \Gamma_{\gamma\beta}^{\bar{\rho}}u_{\bar{\rho}}) + \Gamma_{\alpha\bar{\alpha}}^{\bar{\gamma}}(u_{\bar{\gamma}\beta} + \Gamma_{\bar{\gamma}\beta}^\rho u_\rho) + W_{\bar{\alpha}}(\Gamma_{\alpha\beta}^\gamma)u_\gamma\right.$$

$$\left.+ W_{\bar{\alpha}}(\Gamma_{\alpha\beta}^{\bar{\gamma}})u_{\bar{\gamma}} + \Gamma_{\alpha\beta}^\gamma(u_{\bar{\alpha}\gamma} + \Gamma_{\bar{\alpha}\gamma}^\rho u_\rho) + \Gamma_{\alpha\beta}^{\bar{\gamma}}(u_{\bar{\alpha}\bar{\gamma}} + \Gamma_{\bar{\alpha}\bar{\gamma}}^\rho u_\rho + \Gamma_{\bar{\alpha}\bar{\gamma}}^{\bar{\rho}}u_{\bar{\rho}}) + \Gamma_{\bar{\gamma}\gamma}^\alpha u_{\alpha\beta} - \Gamma_{\bar{\alpha}\bar{\gamma}}^{\bar{\beta}}u_{\alpha\gamma}, u_\beta\right)$$

$$- \sum_{\alpha,\beta,\gamma}\left(\Gamma_{\bar{\alpha}\bar{\gamma}}^\beta u_{\alpha\gamma}, u_{\bar{\beta}}\right)$$

$$= -\sum_{\alpha,\beta}\left(W_\alpha(u_{\bar{\alpha}\beta}), u_\beta\right) + 2ni\sum_\beta\left(T(u_\beta), u_\beta\right) + \Sigma_1 + \Sigma_2 + \Sigma_3, \quad (5.4)$$

by using (3.3) repeatedly, where Σ_1, Σ_2 and Σ_3 denote the summation of terms of type $(*u_\rho, u_\beta)$, $(*u_{\bar\rho}, u_\beta)$ and $(*u_{ab}, u_c)$ respectively. We see that

$$
\Sigma_1 = \sum_{\alpha,\beta,\gamma,\rho} \left((W_{\bar\alpha}\Gamma^\rho_{\alpha\beta} - W_\alpha\Gamma^\rho_{\bar\alpha\beta} - \Gamma^\gamma_{\bar\alpha\beta}\Gamma^\rho_{\alpha\gamma} + \Gamma^\gamma_{\alpha\beta}\Gamma^\rho_{\bar\alpha\gamma} - \Gamma^\gamma_{\bar\alpha\alpha}\Gamma^\rho_{\gamma\beta} + \Gamma^{\bar\gamma}_{\alpha\bar\alpha}\Gamma^\rho_{\bar\gamma\beta} + \Gamma^{\bar\gamma}_{\alpha\beta}\Gamma^\rho_{\bar\alpha\bar\gamma})u_\rho, u_\beta \right)
$$
$$
= (R_{\beta\ \bar\alpha\alpha}^{\ \rho}u_\rho, u_\beta) - 2ni\sum_{\beta,\rho} (\Gamma^\rho_{0\beta}u_\rho, u_\beta), \tag{5.5}
$$

by

$$
R_{\beta\ \bar\alpha\alpha}^{\ \rho} = W_{\bar\alpha}\Gamma^\rho_{\alpha\beta} - W_\alpha\Gamma^\rho_{\bar\alpha\beta} - \Gamma^\gamma_{\bar\alpha\alpha}\Gamma^\rho_{\gamma\beta} + \Gamma^{\bar\gamma}_{\alpha\bar\alpha}\Gamma^\rho_{\bar\gamma\beta} - \Gamma^\gamma_{\bar\alpha\beta}\Gamma^\rho_{\alpha\gamma} + \Gamma^\gamma_{\alpha\beta}\Gamma^\rho_{\bar\alpha\gamma} + \Gamma^{\bar\gamma}_{\alpha\beta}\Gamma^\rho_{\bar\alpha\bar\gamma} + 2ni\Gamma^\rho_{0\beta}, \tag{5.6}
$$

by (2.16), $\Gamma^\gamma_{\alpha\bar\beta} = 0$ in (2.7) and $J_{\bar\alpha\alpha} = \sum_\alpha h(W_{\bar\alpha}, JW_\alpha) = i\sum_\alpha \delta_{\alpha\bar\alpha} = ni$; and

$$
\Sigma_2 = -\frac{i}{2}\sum_{\alpha,\beta,\gamma,\rho} \left((W_{\bar\alpha}Q^{\bar\rho}_{\beta\alpha} - \Gamma^\gamma_{\bar\alpha\beta}Q^{\bar\rho}_{\gamma\alpha} - \Gamma^\gamma_{\bar\alpha\alpha}Q^{\bar\rho}_{\beta\gamma} + \Gamma^{\bar\rho}_{\bar\alpha\bar\gamma}Q^{\bar\gamma}_{\beta\alpha})u_{\bar\rho}, u_\beta \right) = -\frac{i}{2}\sum_{\alpha,\beta,\rho}(Q^{\bar\rho}_{\beta\alpha,\bar\alpha}u_{\bar\rho}, u_\beta) = 0,
$$
$$\tag{5.7}$$

by $\sum_{\alpha,\beta,\rho}(Q^{\bar\rho}_{\beta\alpha,\bar\alpha}u_{\bar\rho}, u_\beta) = \int_M Q^{\bar\rho}_{\beta\alpha,\bar\alpha}u_{\bar\rho}u_{\bar\beta}dV = -\int_M Q^{\bar\beta}_{\rho\alpha,\bar\alpha}u_{\bar\rho}u_{\bar\beta}dV = -\sum_{\alpha,\beta,\rho}(Q^{\bar\beta}_{\rho\alpha,\bar\alpha}u_{\bar\rho}, u_\beta)$ by (2.11);
and

$$
\Sigma_3 = \sum_{\alpha,\beta,\gamma} \left(-\Gamma^\gamma_{\bar\alpha\beta}u_{\alpha\gamma} - \Gamma^\gamma_{\bar\alpha\alpha}u_{\gamma\beta} + \Gamma^{\bar\gamma}_{\alpha\bar\alpha}u_{\bar\gamma\beta} + \Gamma^\gamma_{\alpha\beta}u_{\bar\alpha\gamma} + \Gamma^{\bar\gamma}_{\alpha\beta}u_{\bar\alpha\bar\gamma} + \Gamma^\alpha_{\gamma\gamma}u_{\alpha\beta} - \Gamma^{\bar\beta}_{\bar\alpha\bar\gamma}u_{\alpha\gamma}, u_\beta \right)
$$
$$
- \sum_{\alpha,\beta,\gamma}(\Gamma^\beta_{\bar\alpha\bar\gamma}u_{\alpha\gamma}, u_{\bar\beta})
$$
$$
= \sum_{\alpha,\beta,\gamma}(\Gamma^{\bar\gamma}_{\alpha\bar\alpha}u_{\bar\gamma\beta} + \Gamma^\gamma_{\alpha\beta}u_{\bar\alpha\gamma}, u_\beta) + \sum_{\alpha,\beta,\gamma}(\Gamma^{\bar\gamma}_{\alpha\beta}u_{\bar\alpha\bar\gamma}, u_\beta) - \sum_{\alpha,\beta,\gamma}(\Gamma^\beta_{\bar\alpha\bar\gamma}u_{\alpha\gamma}, u_{\bar\beta}), \tag{5.8}
$$

by using (2.11). We also have

$$
-\sum_{\alpha,\beta}(W_\alpha(u_{\bar\alpha\beta}), u_\beta) = -\sum_{\alpha,\beta}(u_{\bar\alpha\beta}, W^*_\alpha u_\beta) = \sum_{\alpha,\beta}\left(u_{\bar\alpha\beta}, W_{\bar\alpha}(u_\beta) - \Gamma^\alpha_{\bar\gamma\gamma}u_\beta \right)
$$
$$
= \sum_{\alpha,\beta}(u_{\bar\alpha\beta}, u_{\bar\alpha\beta}) + \sum_{\alpha,\beta,\gamma}\left(u_{\bar\alpha\beta}, \Gamma^\gamma_{\bar\alpha\beta}u_\gamma - \Gamma^\alpha_{\bar\gamma\gamma}u_\beta \right) = \sum_{\alpha,\beta}(u_{\bar\alpha\beta}, u_{\bar\alpha\beta}) + \Sigma_4,
$$
$$\tag{5.9}$$

by Lemma 3.1 and (3.3), where $\Sigma_4 = \sum_{\alpha,\beta,\gamma}(\Gamma^{\bar\beta}_{\alpha\bar\gamma}u_{\bar\alpha\gamma} - \Gamma^{\bar\alpha}_{\bar\gamma\gamma}u_{\bar\alpha\beta}, u_\beta)$; and

$$
\sum_\alpha (T(u_\alpha), u_\alpha) - \sum_{\alpha,\rho}(\Gamma^\rho_{0\alpha}u_\rho, u_\alpha) = \sum_\alpha(u_{0\alpha}, u_\alpha). \tag{5.10}
$$

holds by (3.3). Note that the first summation in the right hand side of (5.8) is exactly $-\Sigma_4$ by (2.11). Now substituting (5.5)-(5.10) to (5.4), we get $\int_M u_{\alpha\beta}u_{\bar\alpha\bar\beta}dV$ equals to

$$
\sum_{\alpha,\beta}(u_{\bar\alpha\beta}, u_{\bar\alpha\beta}) + 2ni\sum_\alpha(u_{0\alpha}, u_\alpha) - \sum_{\alpha,\beta}(R_{\alpha\bar\beta}u_\beta, u_\alpha) - \sum_{\alpha,\beta,\rho}(\Gamma^{\bar\beta}_{\alpha\gamma}u_{\bar\alpha\bar\gamma}, u_\beta) - \sum_{\alpha,\beta,\rho}(\Gamma^\beta_{\bar\alpha\bar\gamma}u_{\alpha\gamma}, u_{\bar\beta}),
$$

by using (2.11) for $\sum\limits_{\alpha,\beta,\gamma}\left(\Gamma^{\bar{\gamma}}_{\alpha\beta}u_{\bar{\alpha}\bar{\gamma}},u_\beta\right)=-\sum\limits_{\alpha,\beta,\rho}\left(\Gamma^{\bar{\beta}}_{\alpha\gamma}u_{\bar{\alpha}\bar{\gamma}},u_\beta\right)$ and by (2.17) and (4.7) for $R_{\beta}{}^{\rho}{}_{\bar{\alpha}\alpha}=$
$-R_{\beta}{}^{\rho}{}_{\alpha\bar{\alpha}}=-R_{\beta\bar{\rho}\alpha\bar{\alpha}}=-R_{\alpha\bar{\rho}\beta\bar{\alpha}}=-R_{\alpha}{}^{\rho}{}_{\beta\bar{\alpha}}=-R_{\beta\bar{\rho}}$. So we get

$$i\int_M u_{0\alpha}u_{\bar{\alpha}}dV=\frac{1}{2n}\int_M\left(u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}}-u_{\alpha\bar{\beta}}u_{\bar{\alpha}\beta}+R_{\alpha\bar{\beta}}u_{\bar{\alpha}}u_\beta+\Gamma^{\bar{\beta}}_{\alpha\gamma}u_{\bar{\alpha}\bar{\gamma}}u_{\bar{\beta}}+\Gamma^{\beta}_{\bar{\alpha}\bar{\gamma}}u_{\alpha\gamma}u_\beta\right)dV,$$

The sum of this identity and its conjugation gives

$$i\int_M u_{0\bar{\alpha}}u_\alpha dV-i\int_M u_{0\alpha}u_{\bar{\alpha}}dV$$
$$=\frac{1}{n}\int_M\left(u_{\alpha\bar{\beta}}u_{\bar{\alpha}\beta}-u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}}-R_{\alpha\bar{\beta}}u_{\bar{\alpha}}u_\beta-\Gamma^{\bar{\beta}}_{\alpha\gamma}u_{\bar{\alpha}\bar{\gamma}}u_{\bar{\beta}}-\Gamma^{\beta}_{\bar{\alpha}\bar{\gamma}}u_{\alpha\gamma}u_\beta\right)dV. \qquad (5.11)$$

The result follows.

Comparing with the pseudohermitian case (cf. (9.37) in [10]), the integral formula (5.11) has the extra term $\int_M\left(\Gamma^{\bar{\beta}}_{\alpha\gamma}u_{\bar{\alpha}\bar{\gamma}}u_{\bar{\beta}}+\Gamma^{\beta}_{\bar{\alpha}\bar{\gamma}}u_{\alpha\gamma}u_\beta\right)dV$, which is zero by $\Gamma^{\bar{\beta}}_{\alpha\gamma}=-\frac{i}{2}Q^{\bar{\beta}}_{\gamma\alpha}=0$ in the pseudohermitian case. Note that this extra term is the integral involving second-order covariant derivatives. We can transform them into the integral only involving first-order covariant derivatives via integration by parts in the following lemma.

Lemma 5.2.

$$\int_M\Gamma^{\beta}_{\bar{\alpha}\bar{\gamma}}u_{\alpha\gamma}u_\beta dV=\int_M\left(\frac{i}{2}Q^{\alpha}_{\bar{\gamma}\bar{\beta},\alpha}u_\gamma u_\beta+\frac{1}{4}Q^{\bar{\rho}}_{\alpha\gamma}Q^{\rho}_{\bar{\beta}\bar{\gamma}}u_{\bar{\alpha}}u_\beta-\frac{1}{2}Q^{\bar{\rho}}_{\alpha\gamma}Q^{\gamma}_{\bar{\beta}\bar{\rho}}u_{\bar{\alpha}}u_\beta\right)dV. \qquad (5.12)$$

Proof. By (3.3), (3.11) and (3.14), we have

$$\int_M\Gamma^{\beta}_{\bar{\alpha}\bar{\gamma}}u_{\alpha\gamma}u_\beta dV=\frac{i}{2}\int_M Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_{\alpha\gamma}u_\beta dV=\frac{i}{2}\int_M Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_{\gamma\alpha}u_\beta dV$$
$$=\frac{i}{2}\int_M\left(W_\gamma u_\alpha-\Gamma^{\rho}_{\gamma\alpha}u_\rho+\frac{i}{2}Q^{\bar{\rho}}_{\alpha\gamma}u_{\bar{\rho}}\right)Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\beta dV$$
$$=\frac{i}{2}\int_M\left(u_\alpha(-W_\gamma+\Gamma^{\bar{\gamma}}_{\rho\bar{\rho}})(Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\beta)+\mathscr{E}\right)dV$$
$$=\frac{i}{2}\int_M\left(-W_\gamma(Q^{\beta}_{\bar{\gamma}\bar{\alpha}})u_\alpha u_\beta-Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\alpha W_\gamma(u_\beta)+\Gamma^{\bar{\gamma}}_{\rho\bar{\rho}}Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\alpha u_\beta+\mathscr{E}\right)dV$$
$$=\frac{i}{2}\int_M\left(-W_\gamma(Q^{\beta}_{\bar{\gamma}\bar{\alpha}})u_\alpha u_\beta-Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\alpha\left(u_{\gamma\beta}+\Gamma^{\rho}_{\gamma\beta}u_\rho-\frac{i}{2}Q^{\bar{\rho}}_{\beta\gamma}u_{\bar{\rho}}\right)+\Gamma^{\bar{\gamma}}_{\rho\bar{\rho}}Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\alpha u_\beta+\mathscr{E}\right)dV,$$

where $\mathscr{E}=-\Gamma^{\rho}_{\gamma\alpha}Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\beta u_\rho+\frac{i}{2}Q^{\bar{\rho}}_{\alpha\gamma}Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\beta u_{\bar{\rho}}=\Gamma^{\bar{\alpha}}_{\bar{\gamma}\bar{\rho}}Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\beta u_\rho+\frac{i}{2}Q^{\bar{\rho}}_{\alpha\gamma}Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_\beta u_{\bar{\rho}}$ by (2.11). By

$$2Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_{\gamma\beta}=Q^{\beta}_{\bar{\gamma}\bar{\alpha}}u_{\gamma\beta}-Q^{\gamma}_{\bar{\beta}\bar{\alpha}}u_{\beta\gamma}=0,$$

by (2.11) and (3.14), it equals to

$$\frac{i}{2}\int_M\left(-W_\gamma Q^{\beta}_{\bar{\gamma}\bar{\alpha}}-\Gamma^{\beta}_{\gamma\rho}Q^{\rho}_{\bar{\gamma}\bar{\alpha}}+\Gamma^{\bar{\rho}}_{\gamma\bar{\gamma}}Q^{\beta}_{\bar{\rho}\bar{\alpha}}+\Gamma^{\bar{\rho}}_{\gamma\bar{\alpha}}Q^{\beta}_{\bar{\gamma}\bar{\rho}}\right)u_\alpha u_\beta dV-\frac{1}{4}\int_M\left(Q^{\bar{\beta}}_{\rho\gamma}Q^{\rho}_{\bar{\gamma}\bar{\alpha}}+Q^{\bar{\beta}}_{\rho\gamma}Q^{\alpha}_{\bar{\gamma}\bar{\rho}}\right)u_\alpha u_{\bar{\beta}}dV$$
$$=-\frac{i}{2}\int_M Q^{\beta}_{\bar{\gamma}\bar{\alpha},\gamma}u_\alpha u_\beta dV-\frac{1}{4}\int_M\left(Q^{\bar{\beta}}_{\rho\gamma}Q^{\rho}_{\bar{\gamma}\bar{\alpha}}+Q^{\bar{\beta}}_{\rho\gamma}Q^{\alpha}_{\bar{\gamma}\bar{\rho}}\right)u_\alpha u_{\bar{\beta}}dV.$$

Then (5.12) follows from the last identity, $Q^{\rho}_{\bar{\gamma}\bar{\alpha}}=Q^{\alpha}_{\bar{\gamma}\bar{\rho}}-Q^{\alpha}_{\bar{\rho}\bar{\gamma}}$ and $Q^{\bar{\gamma}}_{\beta\alpha}=-Q^{\bar{\beta}}_{\gamma\alpha}$ in (2.11). $\square$

Substituting (5.12) and its conjugation to (5.11), we get (5.1).

5.2. **The proof of** (5.2).

Note that

$$\int_M (\Delta_b u)^2 dV = \int_M \left(\sum_\alpha u_{\alpha\bar\alpha} + u_{\bar\alpha\alpha} \right)^2 dV = \int_M \sum_{\alpha,\beta} (u_{\alpha\bar\alpha} u_{\beta\bar\beta} + u_{\alpha\bar\alpha} u_{\bar\beta\beta} + u_{\bar\alpha\alpha} u_{\beta\bar\beta} + u_{\bar\alpha\alpha} u_{\bar\beta\beta}) dV$$

$$= \int_M \left(2\left| \sum_\alpha u_{\alpha\bar\alpha} \right|^2 + \sum_{\alpha,\beta} (u_{\bar\alpha\alpha} + 2ih_{\alpha\bar\alpha} u_0) u_{\beta\bar\beta} + \sum_{\alpha,\beta} (u_{\alpha\bar\alpha} - 2ih_{\alpha\bar\alpha} u_0) u_{\bar\beta\beta} \right) dV$$

$$= \int_M \left(4\left| \sum_\alpha u_{\alpha\bar\alpha} \right|^2 + 2ni \sum_\alpha (u_0 u_{\alpha\bar\alpha} - u_0 u_{\bar\alpha\alpha}) \right) dV. \tag{5.13}$$

On the other hand, we have

$$\int_M \sum_\alpha u_0 (u_{\alpha\bar\alpha} - u_{\bar\alpha\alpha}) dV = \sum_\alpha (u_0, u_{\bar\alpha\alpha} - u_{\alpha\bar\alpha}) = \sum_{\alpha,\gamma} (u_0, W_{\bar\alpha}(u_\alpha) - \Gamma^\gamma_{\bar\alpha\alpha} u_\gamma - W_\alpha(u_{\bar\alpha}) + \Gamma^{\bar\gamma}_{\alpha\bar\alpha} u_{\bar\gamma})$$

$$= \sum_\alpha \left(W^*_{\bar\alpha} u_0, u_\alpha \right) - \sum_{\alpha,\gamma} \left(\Gamma^{\bar\gamma}_{\alpha\bar\alpha} u_0, u_\gamma \right) - \sum_\alpha \left(W^*_\alpha u_0, u_{\bar\alpha} \right) + \sum_{\alpha,\gamma} \left(\Gamma^\gamma_{\bar\alpha\alpha} u_0, u_{\bar\gamma} \right)$$

$$= - \sum_\alpha \left(W_\alpha(u_0), u_\alpha \right) + \sum_{\alpha,\gamma} \left(\Gamma^{\bar\alpha}_{\gamma\bar\gamma} u_0, u_\alpha \right) - \left(\Gamma^{\bar\gamma}_{\alpha\bar\alpha} u_0, u_\gamma \right)$$

$$+ \sum_\alpha \left(W_{\bar\alpha}(u_0), u_{\bar\alpha} \right) - \sum_{\alpha,\gamma} \left(\Gamma^\alpha_{\bar\gamma\gamma} u_0, u_{\bar\alpha} \right) + \left(\Gamma^\gamma_{\bar\alpha\alpha} u_0, u_{\bar\gamma} \right)$$

$$= - \sum_\alpha \left(W_\alpha(u_0), u_\alpha \right) + \sum_\alpha \left(W_{\bar\alpha}(u_0), u_{\bar\alpha} \right) = - \sum_\alpha \left(u_{\alpha 0}, u_\alpha \right) + \sum_\alpha \left(u_{\bar\alpha 0}, u_{\bar\alpha} \right)$$

$$= - \sum_{\alpha,\beta} \left(u_{0\alpha} + A_{\alpha\beta} u_{\bar\beta}, u_\alpha \right) + \sum_{\alpha,\beta} \left(u_{0\bar\alpha} + A_{\bar\alpha\bar\beta} u_\beta, u_{\bar\alpha} \right)$$

$$= \int_M \sum_\alpha \left(u_\alpha u_{0\bar\alpha} - u_{\bar\alpha} u_{0\alpha} \right) dV + \int_M \sum_{\alpha,\beta} \left(A_{\bar\alpha\bar\beta} u_\alpha u_\beta - A_{\alpha\beta} u_{\bar\alpha} u_{\bar\beta} \right) dV.$$

Substitute this identity into (5.13) to get

$$\int_M \left((\Delta_b u)^2 - 4\Big| \sum_\alpha u_{\alpha\bar\alpha} \Big|^2 \right) dV = 2ni \sum_{\alpha,\beta} \int_M \left((u_\alpha u_{0\bar\alpha} - u_{\bar\alpha} u_{0\alpha}) + (A_{\bar\alpha\bar\beta} u_\alpha u_\beta - A_{\alpha\beta} u_{\bar\alpha} u_{\bar\beta}) \right) dV,$$

which is equivalent to (5.2).

6. The Proof of Theorem 1.1

Lemma 6.1. *For any $u \in C_0^\infty(M)$, we have*

$$\int_M \left(u_\alpha (\Delta_b u)_{\bar\alpha} + u_{\bar\alpha} (\Delta_b u)_\alpha \right) dV = - \int_M (\Delta_b u)^2 dV.$$

Proof. By (3.13), (3.6) and Lemma 3.1, we get

$$\int_M u_{\bar\alpha} (\Delta_b u)_\alpha dV = \int_M W_\alpha (\Delta_b u) u_{\bar\alpha} dV = (W_\alpha(\Delta_b u), u_\alpha) = \left(\Delta_b u, \left(-W_{\bar\alpha} + \Gamma^\alpha_{\bar\beta\beta} \right) u_\alpha \right)$$

$$= (\Delta_b u, -W_{\bar\alpha}(u_\alpha) + \Gamma^\beta_{\bar\alpha\alpha} u_\beta) = -(\Delta_b u, u_{\bar\alpha\alpha}),$$

by using (3.3). The summation of this identity and its conjugation gives the result. $\qquad\square$

Note that

$$\int_M (\Delta_b f)dV = 0,$$

holds for any $f \in C_0^\infty(M)$. Apply $f = \|\partial_b u\|^2 \in C_0^\infty(M)$ to this identity for $u \in C_0^\infty(M)$. By the Bochner-type formula (4.1) and Lemma 6.1, we get

$$
\begin{aligned}
0 = \int_M \Bigg(& u_{\alpha\beta}u_{\bar\alpha\bar\beta} + u_{\alpha\bar\beta}u_{\bar\alpha\beta} + 2i(u_\alpha u_{0\bar\alpha} - u_{\bar\alpha}u_{0\alpha}) + ni(A_{\bar\alpha\bar\beta}u_\alpha u_\beta - A_{\alpha\beta}u_{\bar\alpha}u_{\bar\beta}) + R_{\alpha\bar\beta}u_{\bar\alpha}u_\beta \\
& - \frac{1}{2}(\Delta_b u)^2 + \frac{i}{2}Q^\gamma_{\bar\alpha\bar\beta,\gamma}u_\alpha u_\beta - \frac{i}{2}Q^{\bar\gamma}_{\alpha\beta,\bar\gamma}u_{\bar\alpha}u_{\bar\beta} - \frac{1}{4}Q^{\bar\rho}_{\alpha\gamma}Q^\rho_{\bar\beta\bar\gamma}u_{\bar\alpha}u_\beta \Bigg)dV. \quad (6.1)
\end{aligned}
$$

Apply (5.1) and (5.2) to get

$$
\begin{aligned}
& i\int_M (u_\alpha u_{0\bar\alpha} - u_{\bar\alpha}u_{0\alpha})dV = Ci\int_M (u_\alpha u_{0\bar\alpha} - u_{\bar\alpha}u_{0\alpha})dV + (1-C)i\int_M (u_\alpha u_{0\bar\alpha} - u_{\bar\alpha}u_{0\alpha})dV \\
& = \int_M \Bigg(\frac{C}{n}u_{\alpha\bar\beta}u_{\bar\alpha\beta} - \frac{C}{n}u_{\alpha\beta}u_{\bar\alpha\bar\beta} - \frac{C}{n}R_{\alpha\bar\beta}u_{\bar\alpha}u_\beta \\
& \quad - \frac{C}{2n}iQ^\gamma_{\bar\alpha\bar\beta,\gamma}u_\alpha u_\beta + \frac{C}{2n}iQ^{\bar\gamma}_{\alpha\beta,\bar\gamma}u_{\bar\alpha}u_{\bar\beta} - \frac{C}{2n}Q^{\bar\rho}_{\alpha\gamma}Q^\rho_{\bar\beta\bar\gamma}u_{\bar\alpha}u_\beta + \frac{C}{n}Q^{\bar\rho}_{\alpha\gamma}Q^\gamma_{\bar\beta\bar\rho}u_{\bar\alpha}u_\beta \\
& \quad - \frac{2(1-C)}{n}\Bigg| \sum_\alpha u_{\alpha\bar\alpha} \Bigg|^2 + \frac{1-C}{2n}(\Delta_b u)^2 + (1-C)iA_{\alpha\beta}u_{\bar\alpha}u_{\bar\beta} - (1-C)iA_{\bar\alpha\bar\beta}u_\alpha u_\beta \Bigg)dV. \quad (6.2)
\end{aligned}
$$

Substituting (6.2) to (6.1), we get

$$
\begin{aligned}
0 = \int_M \Bigg\{ & \left(1 + \frac{2C}{n}\right)u_{\alpha\bar\beta}u_{\bar\alpha\beta} + \left(1 - \frac{2C}{n}\right)u_{\alpha\beta}u_{\bar\alpha\bar\beta} - \frac{4(1-C)}{n}\Bigg| \sum_\alpha u_{\alpha\bar\alpha} \Bigg|^2 \\
& + \left(-\frac{1}{2} + \frac{1-C}{n}\right)(\Delta_b u)^2 + \left(1 - \frac{2C}{n}\right)R_{\alpha\bar\beta}u_{\bar\alpha}u_\beta + i\left(n - 2(1-C)\right)\left(A_{\bar\alpha\bar\beta}u_\alpha u_\beta - A_{\alpha\beta}u_{\bar\alpha}u_{\bar\beta}\right) \\
& + i\left(\frac{1}{2} - \frac{C}{n}\right)\left(Q^\gamma_{\bar\alpha\bar\beta,\gamma}u_\alpha u_\beta - Q^{\bar\gamma}_{\alpha\beta,\bar\gamma}u_{\bar\alpha}u_{\bar\beta}\right) - \left(\frac{1}{4} + \frac{C}{n}\right)Q^{\bar\rho}_{\alpha\gamma}Q^\rho_{\bar\beta\bar\gamma}u_{\bar\alpha}u_\beta + \frac{2C}{n}Q^{\bar\rho}_{\alpha\gamma}Q^\gamma_{\bar\beta\bar\rho}u_{\bar\alpha}u_\beta \Bigg\}dV. \\
& \quad\quad (6.3)
\end{aligned}
$$

Let $\left(1 + \frac{2C}{n}\right)\frac{1}{n} - \frac{4(1-C)}{n} = 0$, i.e., $C = \frac{3n}{4n+2}$ in (6.3). By using $u_{\alpha\bar\beta}u_{\bar\alpha\beta} = \sum_{\alpha,\beta} |u_{\alpha\bar\beta}|^2 \geq \frac{1}{n}|\sum_\alpha u_{\alpha\bar\alpha}|^2$ (cf. p. 489 in [19]), we get

$$
\begin{aligned}
0 \geq \int_M & \frac{2(n-1)}{2n+1}u_{\alpha\beta}u_{\bar\alpha\bar\beta} - \frac{n^2-1}{n(2n+1)}(\Delta_b u)^2 + \frac{2(n-1)}{2n+1}R_{\alpha\bar\beta}u_{\bar\alpha}u_\beta + i\frac{2n^2-2}{2n+1}\left(A_{\bar\alpha\bar\beta}u_\alpha u_\beta - A_{\alpha\beta}u_{\bar\alpha}u_{\bar\beta}\right) \\
& + \frac{n-1}{2n+1}i\left(Q^\gamma_{\bar\alpha\bar\beta,\gamma}u_\alpha u_\beta - Q^{\bar\gamma}_{\alpha\beta,\bar\gamma}u_{\bar\alpha}u_{\bar\beta}\right) - \frac{2n+7}{8n+4}Q^{\bar\rho}_{\alpha\gamma}Q^\rho_{\bar\beta\bar\gamma}u_{\bar\alpha}u_\beta + \frac{3}{2n+1}Q^{\bar\rho}_{\alpha\gamma}Q^\gamma_{\bar\beta\bar\rho}u_{\bar\alpha}u_\beta \Bigg\}dV \\
= \int_M \Bigg\{ & \frac{2(n-1)}{2n+1}u_{\alpha\beta}u_{\bar\alpha\bar\beta} + \frac{n^2-1}{n(2n+1)}\lambda_1 u(\Delta_b u) \\
& + \frac{2(n-1)}{2n+1}\Bigg(R_{\alpha\bar\beta}u_{\bar\alpha}u_\beta + i(n+1)\left(A_{\bar\alpha\bar\beta}u_\alpha u_\beta - A_{\alpha\beta}u_{\bar\alpha}u_{\bar\beta}\right) \\
& + \frac{i}{2}\left(Q^\gamma_{\bar\alpha\bar\beta,\gamma}u_\alpha u_\beta - Q^{\bar\gamma}_{\alpha\beta,\bar\gamma}u_{\bar\alpha}u_{\bar\beta}\right) - \frac{2n+7}{8(n-1)}Q^{\bar\rho}_{\alpha\gamma}Q^\rho_{\bar\beta\bar\gamma}u_{\bar\alpha}u_\beta + \frac{3}{2(n-1)}Q^{\bar\rho}_{\alpha\gamma}Q^\gamma_{\bar\beta\bar\rho}u_{\bar\alpha}u_\beta \Bigg) \Bigg\}dV, \\
& \quad\quad (6.4)
\end{aligned}
$$

for $\Delta_b u = -\lambda_1 u$. We use the following lemma to handle the second term in the right hand side.

Lemma 6.2.
$$-2 \int_M \sum_\alpha |u_\alpha|^2 dV = \int_M u(\Delta_b u) dV.$$

Proof. By Lemma 3.1 and (3.3), we have

$$\int_M \sum_\alpha |u_\alpha|^2 dV = \sum_\alpha (u_\alpha, u_\alpha) = \sum_\alpha (W_\alpha u, u_\alpha) = \sum_\alpha (u, W_\alpha^* u_\alpha)$$

$$= -\sum_\alpha (u, W_{\bar\alpha}(u_\alpha)) + \sum_{\alpha,\beta} (u, \Gamma_{\bar\beta\beta}^\alpha u_\alpha)$$

$$= -\sum_\alpha (u, u_{\bar\alpha\alpha}) - \sum_{\alpha,\beta} (u, \Gamma_{\bar\alpha\alpha}^\beta u_\beta) + (u, \Gamma_{\bar\beta\beta}^\alpha u_\alpha) = -\sum_\alpha (u, u_{\bar\alpha\alpha}).$$

The summation of this identity and its conjugation gives the result. $\qquad\square$

Let $X = u_{\bar\alpha} W_\alpha \in T^{(1,0)} M$. If (1.6) holds, by the definition of Ricci tensor, and Tor given by (2.13) and Q_1, Q_2, Q_3 given by (2.9), we get

$$R_{\alpha\bar\beta} u_{\bar\alpha} u_\beta + i(n+1)\left(A_{\bar\alpha\bar\beta} u_\alpha u_\beta - A_{\alpha\beta} u_{\bar\alpha} u_{\bar\beta}\right)$$
$$+ \frac{i}{2}\left(Q_{\bar\alpha\bar\beta,\gamma}^\gamma u_\alpha u_\beta - Q_{\alpha\beta,\bar\gamma}^{\bar\gamma} u_{\bar\alpha} u_{\bar\beta}\right) - \frac{2n+7}{8(n-1)} Q_{\alpha\gamma}^{\bar\rho} Q_{\bar\beta\bar\gamma}^\rho u_{\bar\alpha} u_\beta + \frac{3}{2(n-1)} Q_{\alpha\gamma}^{\bar\rho} Q_{\bar\beta\bar\rho}^\gamma u_{\bar\alpha} u_\beta \geq \kappa \sum_\alpha |u_\alpha|^2.$$
$$(6.5)$$

So by Lemma 6.2, (6.4) and (6.5), we get

$$0 \geq \int_M \left(-\frac{2(n^2-1)}{n(2n+1)}\lambda_1 + \frac{2(n-1)}{2n+1}\kappa\right) \sum_\alpha |u_\alpha|^2 dV, \qquad (6.6)$$

i.e. $\lambda_1 \geq \frac{\kappa n}{n+1}$. Theorem 1.2 is proved.

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